THE

# NATURE AND LAWS
OF
# CHANCE.

CONTAINING, AMONG OTHER PARTICULARS,

The Solutions of ſeveral abſtruſe and important Problems.

The Doctrine of Combinations and Permutations clearly deduced.

A new and comprehenſive Problem of great Uſe in diſcovering the Advantage or Loſs in Lotteries. Raffles, &c.

A curious and extenſive Problem on the Duration of Play.

Problems for determining the Probability of winning at Bowls, Quoits, Cards, &c.

A Problem for finding the Trials wherein it may be undertaken that a propoſed Event ſhall happen or fail a given Number of Times.

A Problem to find the Chance for a given Number of Points on a given Number of Dice.

Full and clear Inveſtigations of two Problems of Mr. *De Moivre*'s; one of them allowed by that Author to be the moſt uſeful on the Subject, but their Demonſtrations omitted by him.

Two new Methods for ſumming of Series.

THE WHOLE

*After a new, general, and conſpicuous Manner,*

AND ILLUSTRATED WITH

A GREAT VARIETY OF EXAMPLES.

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By THOMAS SIMPSON, F. R. S.

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A NEW EDITION.

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# PREFACE.

AS it is reaſonable to expect that one, who undertakes to lay his thoughts before the public on any ſubject, ſhould be able to point out ſome advantage to be reaped from it, it may not be improper, firſt to ſhew upon what ground I was induced to hope that the few following ſheets might be of ſervice, and claim a reception. And here I muſt ingenuouſly own, that what had the greateſt weight was not any immoderate opinion of my own productions, but the ſcarcity and high price of ſome, and the imperfection of the reſt of the few books at preſent among us on the ſubject; moſt of theſe being wholly taken up with a few low problems, and, what is ſtill a greater defect, without affording any general principles whereby the thing might be extended farther. There is indeed but one, that I have met with, entirely free from this objection; and, though it neither wants matter nor elegance to recommend it, yet the price muſt, I am ſenſible, have put it out of the power of many to purchaſe it; and even ſome, who want no means to gratify their deſires this way, and who might not be inclinable to ſubſcribe a guinea for a ſingle book, however excellent, may not ſcruple the beſtowing of a ſmall matter on one, that perhaps may ſerve equally well for their purpoſe.

*I am ſatisfied, it may be deemed a ſort of preſumption to attempt, upon any account, a ſubject like this, after ſo great a man as* Mr. De Moivre; *and that ſome, who would paſs for conſiderable judges, have not been wanting to cenſure and contemn this performance already on that head, before having the leaſt knowledge of the particulars therein contained. But they would do well to reflect, whether to proceed thus, may not prove a greater impeachment of their own judgment and conduct, even among their friends, than of his, whoſe work, without any offence given, they endeavour to depreciate.*

*I have perhaps as high an opinion of that learned author's productions as any of thoſe gentlemen. But does it follow, becauſe one perſon has done well on this ſubject, that nothing further can be neceſſary? Beſides, it is not every one that has a genius fitted for the moſt exalted ſpeculations, or that is capable of reading the works of the moſt ſublime and celebrated authors; and, therefore, though I ſhould go no farther than to bring down ſome of the beſt and moſt uſeful things already known to the level of ordinary capacities, I ſhould think this might, in ſome meaſure, exempt me from cenſure. However, I would not have the reader conclude from hence, that he has nothing more to expect in theſe few* *ſheets than a bare collection of low matters; or that, like ſome of our preſent mathematical writers, I ſhould be poorly ambitious of appearing the author of a performance, that would, was every bird to claim his own feather, be ſtript as naked as the* Jay *in the fable.*

*It is not my intention to ſay a great deal in behalf of this work, knowing that if there be a real merit, it will beſt recommend itſelf, and that without this, it will be in vain for me to hope for a reception: nevertheleſs, it may not be amiſs to point out ſome of the moſt* uſeful *and* curious *propoſitions therein, and make ſuch uſeful obſervations thereon, as may be of ſome ſervice in, or ſhew it not unworthy, the peruſal.*

*The 1ſt is a propoſition of uſe throughout the work, and therefore ought to be well underſtood by a young beginner; towards which the ſix ſucceeding caſes in illuſtration thereof, will not a little contribute, ſince, by them alone, a perſon but tolerably ſkilled in common arithmetick may ſoon arrive to ſome proficiency in the ſubject. In the 2d and 3d problems, beſides other things, the doctrine of* Combinations *and* Permutations *is clearly and fully deduced. Nor is the 5th, ſhewing the* Probability of an aſſigned Event happening a given Number of Times in a given Number of Trials, *for general uſe, inferior to either of them.*

them. *The 6th is very comprehenſive, and of great uſe in lotteries, cards,* &c. *And the 15th, for* finding the Trials wherein one may undertake that a propoſed Event ſhall happen a given number of Times, *has been long looked on as a problem of the greateſt note and conſequence, and is ſolved in a more* general *manner than hitherto. The 16th, 17th, and 20th, are alſo problems of ſome note and difficulty. But the 22d* to find the Chances for a given Number of Points on a given Number of Dice, *and the 25th on the* Duration of Play, *are two of the moſt intricate and remarkable on the ſubject, and both ſolved by methods entirely new. The 27th is a problem that was propoſed to the public ſome time ago in* Latin, *as a very difficult one, and has not (that I know of) been anſwered before. And the 24th and 30th are the ſame with the two new ones, inſerted at the end of Mr.* De Moivre's *laſt edition, whoſe demonſtrations that learned author was pleaſed to reſerve to himſelf, and are here fully and clearly inveſtigated; both which problems are of conſiderable importance. Laſtly, in the* Lemmas, *among other things, will be found two new methods for ſumming of ſeries; and by help of one of them the value of a ſeries of powers, whether whole or broken, is determined in a more* conciſe *and* general *manner than heretofore.*

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# OF

# CHANCES.

## DEFINITION I.

THE *probability* of the happening of an event is to be underſtood as the ratio of the chances by which that event may happen, to all the chances by which it may either happen or fail.

As, ſuppoſing it were required to expreſs the probability of throwing either an ace or duce at the firſt throw with a ſingle die: then there being in all 6 different chances or ways that the die may fall, and only 2 of them for the ace or duce to come upward, the probability of the happening of one of theſe will be $\frac{2}{6}$ or $\frac{1}{3}$, and that of the contrary $\frac{4}{6}$ or $\frac{2}{3}$: or, more generally, ſuppoſing there be $a$ chances for the happening of an event, and $b$ chances for the contrary; then the probability of happening will be $\frac{a}{a+b}$, and that of failing $\frac{b}{a+b} = 1 - \frac{a}{a+b}$.

## COROLLARY.

Hence it appears, that if the probability of the happening of an event be ſubſtracted from unity, the remainder will be the probability of its failing, and *vice verſa*.

## DEFINITION II.

The *expectation* on an event is conſidered as the *preſent certain* value, or worth of whatever ſum or thing is depending on the happening of that event, and is compounded of that ſum and the probability of obtaining it.

## CORALLARY.

Therefore, if the expectation on an event be divided by the value of the thing expected on the happening of that event, the quotient will be the probability of happening.

## EXAMPLE.

Suppoſe *A* to throw once with a ſingle die, on condition that if either an ace or duce comes up he ſhall be intitled to 20 ſhillings; then, becauſe the probability of his receiving the ſaid ſum is $\frac{1}{3}$ (*Def.* I.) $\frac{1}{3} \times 20s.$ or $\frac{1}{3}$ of 20*s.* will be the expectation in this caſe.

## PROBLEM I.

*To find the Probability that two aſſigned Events ſhall both happen.*

Let the probability of the happening of the firſt of the two events be denoted by $\frac{a}{a+b}$, and that of the ſecond by $\frac{c}{c+d}$; and ſuppoſe the happening of both to entitle a perſon *B* to the ſum *S*. Now if the firſt of theſe ſhould happen, it is manifeſt that, from that time till the ſecond is determined, the expectation of *B* will be $\frac{c}{c+d} \times S$, or ſo much is the ſum that he might in that circumſtance receive as an equivalent for his chance of obtaining the ſum $S$. But the probability of getting into this circumſtance, or being intitled to the value $\frac{c}{c+d} \times S$, being only $\frac{a}{a+b}$, his expectation therefore, before either of the events is decided, can be only $\frac{a}{a+b}$, part, of $\frac{c}{c+d} \times S$, or $\frac{a}{a+b} \times \frac{c}{c+d} \times S$, and therefore the required probability of receiving it, or the happening of both the events, only $\frac{a}{a+b} \times \frac{c}{c+d}$; that is, the probability that any two aſſigned events ſhall both happen, will be equal to the product of the probabilities of the happening of thoſe events conſidered ſeparately.

## COROLLARY.

Wherefore, ſince the probability of the happening of each of theſe events may be compounded of the probabilities of the happening of two others, as well as that of receiving the ſum $S$ is of them two, &c. it follows that the probability of the happening of any given number of events, *i. e.* that they ſhall all happen, is equal to the product of all the probabilities of happening of thoſe events conſidered ſingly.

But as this concluſion is the baſis whereon all the ſucceeding calculations are founded, it may not be improper to enlarge a little farther thereon, and endeavour to render the ſame ſtill more

## DEFINITION II.

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## PROBLEM I.

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But as this concluſion is the baſis whereon all the ſucceeding calculations are founded, it may not be improper to enlarge a little farther thereon, and endeavour to render the ſame ſtill more familiar and eaſy by a numerical exegeſis, in order to ſhew thoſe who are not ſo well acquainted with algebraic computation how to reaſon with certainty on the ſubject.

I. Suppoſe *A*, holding a ſingle die, to begin to throw, on condition that if an ace comes up both the firſt and ſecond throws he ſhall receive 1*l*. Now if an ace ſhould come up the firſt throw, the expectation, or worth of his chance would, it is manifeſt, then be $\frac{1}{6}$ of 1*l*. but the probability of getting into that circumſtance being but $\frac{1}{6}$ (*Def.* I.) the required expectation before he begins to throw will be only $\frac{1}{6}$ of $\frac{1}{6}$ of 1*l*. This divided by 1*l*. gives $\frac{1}{6}\times\frac{1}{6}$ for the probability of receiving the ſum propoſed, (*Cor.* to *Def.* II.) equal to the product of the probabilities of happening of two aces, when conſidered ſeparately.

II. *B* upon certain conſiderations agrees to depoſit to *A* the ſum of 50 ſhillings, if in the three firſt throws with a ſingle die the latter throws three aces. If an ace ſhould come up the firſt throw, it is evident, from the laſt caſe, that the expectation of *A* would be $\frac{1}{6}$ of $\frac{1}{6}$ of 50 ſhillings; but as the probability of that happening is only $\frac{1}{6}$, his expectation before the firſt throw will therefore be but $\frac{1}{6}$ of $\frac{1}{6}$ of $\frac{1}{6}$ of 50 ſhillings; and conſequently the probability of his receiving the propoſed ſum $\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}$ ($=\frac{1}{216}$) equal to the product of the probabilities of the happening of the three component events conſidered ſeparately; agreeable to the general corollary.

*Note.* If, inſtead of throwing an ace three times together with a ſingle die, the condition had been to throw three aces at once with three dice, the expectation would have been the very ſame: and it may be farther obſerved, that in divers other caſes, where one event is, or may be conceived to be, compounded of ſeveral others more ſimple, which are all decided at the ſame time, as in throwing of dice, *&c.* it will facilitate the reaſoning very much, to ſuppoſe them determined one by one in a ſucceſſive order.

III. Imagine a heap of 16 counters, whereof 6 are red, and the reſt black; and a perſon to draw out 2 of them blindfold: to find the odds that one or both of thoſe ſhall be red ones. Let them be ſuppoſed to be taken one at a time, and that, if either of them prove red ones, the drawer ſhall be intitled to a given ſum, as 1*l.* If the firſt drawn ſhould happen to be red, his expectation on the ſecond would vaniſh; for as the happening of one red one inſures to him the propoſed ſum, it cannot be of the leaſt advantage to draw another red one afterwards; and therefore the whole expectation on the ſecond counter is compounded of, or depends on, the probability of drawing a black one firſt, and a red one next after. Now the probability of drawing a black one firſt, is $\frac{10}{16}$; and if a black one ſhould be drawn, there being then only 15 counters remaining, the probability of taking a red one next, it is manifeſt, would be $\frac{6}{15}$, and the expectation thereon $\frac{6}{15}$ of 1*l.* But the probability of getting into that circumſtance being only $\frac{10}{16}$, the true expectation on the ſecond counter is therefore only $\frac{10}{16}$ of $\frac{6}{15}$ of 1*l.* which added to $\frac{6}{16}$ of 1*l.* his expectation on the firſt, gives $\frac{5}{8}$ of 1*l.* for the total; wherefore the probability that one or both the counters ſhould be red ones is $\frac{5}{8}$, and the required odds as 5 to 3. But this may be determined with more facility, by firſt finding the probability that neither of the two counters ſhall be red ones; for if the firſt ſhould come out a black one, of which the probability is $\frac{10}{16}$, then there remaining but 15 counters, the probability of taking a black one next would be $\frac{9}{15}$; which drawn into ($\frac{10}{16}$) the foregoing *(See the Cor.)* gives $\frac{3}{8}$ for the probability that both the firſt and ſecond ſhall be black ones; and therefore the odds are 5 to 3, as before found.

IV. Let there be a lottery conſiſting of 100 tickets, wherein there are four prizes: to find the probability that in the three firſt numbers that are drawn there ſhall be one or more prizes. Here, as in the laſt caſe, it will be convenient to find firſt the probability that none of thoſe numbers ſhall be prizes: in order to which, let the firſt be ſuppoſed to have come out a blank; then there being 99 tickets remaining, and 95 of them blanks, the probability of drawing a blank next will, it is manifeſt, be $\frac{95}{99}$; which therefore multiplied by $\frac{96}{100}$, the probability of the firſt coming out as ſuppoſed, yields $\frac{96}{100} \times \frac{95}{99}$ for the probability that the two firſt tickets that are drawn ſhall be both blanks. If both theſe ſhould happen to be ſo, the probability of taking a blank will be $\frac{94}{98}$, which therefore drawn into $\frac{96}{100} \times \frac{95}{99}$ gives ($\frac{96}{100} \times \frac{95}{99} \times \frac{94}{98} = \frac{7144}{8085}$) the probability of drawing all three blanks; this ſubſtracted from unity *(Cor. to Def.* I.) leaves $\frac{941}{8085}$ for the required probability of taking one or more prizes; wherefore the odds that all the three are blanks, are as 7144 to 941, or as 15 to 2 nearly.

V. *B* holding four common dice, bets one guinea to two, that one ace, and no more, comes up the firſt throw: to find his advantage or diſadvantage. Let the dice be conceived to be thrown one by one: then if the firſt ſhould come up an ace, the probability of miſſing the ace all the next three throws being $\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}$ (*Caſe* II.) the probability that the firſt comes up an ace, and all the reſt otherwiſe, is $\frac{1}{6}$ of $\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}$ $=\frac{125}{1296}$. But ſince this is only one of four ways by which *B* may win, or becauſe there is the ſame chance that the 2d, or 3d, or 4th, may come up an ace, and the reſt otherwiſe, $\frac{125}{1296}\times 4$ $=\frac{125}{324}$ will, it is evident, be the probability of *B*'s winning; wherefore his expectation on the three guineas is $\frac{125}{324}$ of that ſum, from which deducting his own ſtake, there remains $\frac{17}{108}$ of a guinea $=$ 3*s*. 3*d*. $\frac{2}{3}$ for the required advantage, or ſo much is the ſum that he might give, upon an equality of chance, to another perſon to lay him the ſame wager.

VI. One with a ſingle die propoſes to throw the ace twice before either the duce or tray comes up once; to find the odds againſt him. There is given the probability that the ace comes up the firſt ſignificant throw, where either ace, duce or tray muſt come up, equal to $\frac{1}{3}$, which is alſo the probability of the ſame thing happening the ſecond ſignificant throw; wherefore the probability that an ace ſhall come up both thoſe throws, it is evident, will be $\frac{1}{3}$ of $\frac{1}{3}$, or $\frac{1}{9}$, and the required odds as 8 to 1.

PROBLEM

## PROBLEM II.

*Let any given Number (n) of Letters, as A, B, C, D, E,* &c. *or Things repreſented by them, be diſpoſed in a regular Order; and let a given Number (p) of them be taken, one by one, as it happens: to find the Probability that they ſhall come out according to the very Order in which they are placed; as A 1ſt, B 2d, C 3d, &c.*

## SOLUTION.

Since there are $n$ letters in all, the probability of drawing $A$ firſt is $\frac{1}{n}$. If $A$ ſhould be ſo drawn, then there remaining $n - 1$ letters, the probability of taking $B$ next will be $\frac{1}{n-1}$, whence *(Cor. to Prob.* I.) it is manifeſt that the probability that $A$ is taken firſt and $B$ next, is $\frac{1}{n} \times \frac{1}{n-1}$. If theſe two ſhould be ſo taken, the probability of drawing $C$ next will be $\frac{1}{n-2}$, becauſe then only $n - 2$ letters will be remaining: wherefore, *(by the ſame Cor.)* $\frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2}$ is the probability of taking the three firſt according to the order propoſed: whence, from the manner of the proceſs, it is evident, that the probability of coming out of the 4, 5, or 6 firſt, *&c.* according to the ſame order, will be $\frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2} \times \frac{1}{n-3}$, $\frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2} \times \frac{1}{n-3} \times \frac{1}{n-4}$, or $\frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2} \times \frac{1}{n-3} \times \frac{1}{n-4} \times \frac{1}{n-5}$, *&c.* reſpectively; and conſequently that the required probability is $\frac{1}{n}\times\frac{1}{n-1}\times\frac{1}{n-2}\times$ $\frac{1}{n-3}$, &c. continued to *p* factors, or $\frac{1}{n\times\overline{n-1}\times\overline{n-2},\&c.}$, whoſe denominator conſiſts of the ſame number of factors.

## COROLLARY.

Hence, if *p* be ſuppoſed = *n*, or all the letters, or things, be to be taken, the above ſaid probability will become $\frac{1}{1\times2\times3\times4,\ \&c.\ \text{to}\ n}$: therefore ſince there is but one chance, or way, for all the letters, or things, to come out in that order, it follows that the whole number of permutations, or all the different ways which the ſame things can poſſibly be taken, will be the denominator of that fraction: or, 1×2×3×4, &c. continued to as many factors as there are things, (*Def.* I.)

## EXAMPLE.

Suppoſe it were required to find the number of changes on 7 bells; then taking the firſt 7 factors of 1×2×3×4, &c. and multiplying them together; there comes out 5040, for the number that was to be found.

## PROBLEM III.

*If out of any given Number (n) of Things, as A, B, C, D, E, F, &c. a Perſon be to take a given Number (p) of them, as it happens; what is the Probability that the Things ſo taken ſhall be the (p) firſt of the foregoing order, as A, B, C, D, &c.*

SOLUTION.

## SOLUTION.

Let the things be taken one by one; then becauſe, in the whole, there are *p* aſſigned, the probability of drawing one of theſe firſt will be $\frac{p}{n}$. If one of thoſe ſhould be drawn, then there remaining $n-1$, of which $p-1$ will be of the aſſigned ones, the probability of drawing one of theſe next would be $\frac{p-1}{n-1}$; wherefore, (*by Cor. to Prob.* I.) $\frac{p}{n}\times\frac{p-1}{n-1}$, will be the probability that the two firſt taken ſhall be both of the aſſigned ones. If both thoſe ſhould be ſo, the probability of taking another of the aſſigned ones next, it is obvious, would be $\frac{p-2}{n-2}$; therefore (*by the ſame*) the probability of all the three firſt being of the aſſigned ones is $\frac{p}{n}\times\frac{p-1}{n-1}\times\frac{p-2}{n-2}$: whence, from the method of the proceſs, it is manifeſt that the probability ſought will be $\frac{p}{n}\times\frac{p-1}{n-1}\times\frac{p-2}{n-1}\times\frac{p-3}{n-3}\times\frac{p-4}{n-4}$, &c. continued to *p* factors; or, which is the ſame, $=\frac{p\times\overline{p-1}\times\overline{p-2}\times\overline{p-3},\text{ \&c.}}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3},\text{ \&c.}}$ where the numerator and denominator conſiſt each of *p* factors: but the factors in the numerator forming an arithmetical progreſſion, whoſe greateſt term is *p*, and common difference 1, and the number of terms being *p*, the leaſt term or factor muſt neceſſarily be $=1$, and the ſaid numerator in an inverted order, $1\times2\times3\times4$, &c. to *p*; and conſequently the probability $= \frac{1\times2\times3\times4}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}$, &c.

## COROLLARY.

If the $p$ things drawn one by one, as above, be mixed again among the reſt; and a ſecond perſon afterwards draws, at random, an equal number, $p$, at once, it is manifeſt, that the probability of his taking the very ſame, or any other, $p$, aſſigned ones, will be $\frac{1\times2\times3\times4}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}$, &c. the ſame as above; and therefore ſince there is here only one chance or way for taking the ſaid $p$ aſſigned things, this one chance muſt be to the whole number of chances, or all the ſeveral ways that $p$ things can be ſo taken in $n$ things, as the num. of the above fraction to its denominator, by *Def.* I. Wherefore the number of combinations, or different ways by which $n$ things may be taken, $p$ by $p$, is equal to $\frac{n\times\overline{n-1}\times\overline{n-2}, \text{ \&c. to } p \text{ factors,}}{1\times2\times3 \quad \text{ \&c. to } p \text{ factors,}}$ or $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$, &c. continued to as many factors as there are things in one combination.

## EXAMPLE.

Let it be required to find how many different pairs, or combinations, of two, can be made with four aces. Here taking $\frac{n}{1}\times\frac{n-1}{2}$ the two firſt factors of the general expreſſion, (becauſe two are the things in one combination) and writing 4 inſtead of $n$, we have $\frac{4}{1}\times\frac{4-1}{2}=6$ for the number that was to be found.

## PROBLEM IV.

*There are ſeveral Sorts of Things, or Letters, as a a a, b b b b b, c c c c, d d d d d d d d, to be mixed promiſcuouſly together, in order to be taken one by one, as it happens: what is the Probability that any given Number of theſe Sorts come out in the very order here exhibited,* i. e. *that all the a's come out firſt, all the b's next, &c. &c?*

## SOLUTION.

Since the number of letters of all ſorts is here 20, and that of the firſt ſort 3; the probability that theſe ſhall be all taken before a letter of any other ſort, is $\frac{1}{20}\times\frac{2}{19}\times\frac{3}{18}$, *(by the laſt prob.)*. If theſe ſhould be ſo taken, the probability of drawing the *b's* next in order, it is manifeſt, would be $\frac{1}{17}\times\frac{2}{16}\times\frac{3}{15}\times\frac{4}{14}\times\frac{5}{13}$ *(by the ſame)*: wherefore the probability that both *a's* and *b's* ſhall be taken in order will be $\frac{1}{20}\times\frac{2}{19}\times\frac{3}{18}\times\frac{1}{17}\times\frac{2}{16}\times\frac{3}{15}\times\frac{4}{14}\times\frac{5}{13}$ *(by Prob.* I.) In like manner the probability of taking the three firſt ſorts in order, will be had $\frac{1}{20}\times\frac{2}{19}\times\frac{3}{18}\times\frac{1}{17}\times\frac{2}{16}\times\frac{3}{15}\times\frac{4}{14}\times\frac{5}{13}\times\frac{1}{12}\times\frac{2}{11}\times\frac{3}{10}\times\frac{4}{9}$; which is alſo the probability of taking all the four ſorts in order, becauſe when the firſt three of them are ſo taken, the laſt muſt follow of courſe.

## COROLLARY.

If the number of letters or things of the firſt ſort be $p$, of the ſecond ſort $q$, of the third $r$, &c.

and if $n$ be the whole number of things of all ſorts: it is evident by inſpection, from the foregoing proceſs, that the probability that any number of thoſe ſorts will come out in the propoſed order is

$$\frac{1\times2\times3,\&c.\text{ to }p,\ \times1\times2\times3,\&c.\text{ to }q,\ \times1\times2\times3,\&c.\text{ to }r,\&c.}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\overline{n-5}\times\overline{n-6}\times\overline{n-7}\times\overline{n-8}\times\overline{n-9}\,\&c.}$$

where the numerator conſiſts of as many ſeries as there are ſorts to be taken in order, and the denominator of as many factors as there are in all thoſe ſeries.

## PROBLEM V.

*Suppoſing the Proportion of the Chances for the Happening of an Event to the Chances for its Failing be as a to b, in any one Trial, and (n) be the Number of Trials: to find the Probability that the Event happens preciſely (p) times in thoſe Trials.*

## SOLUTION.

Since the probability of happening of the event at any one aſſigned trial is $\frac{a}{a+b}$, and that of the contrary $\frac{b}{a+b}$; the probability that it ſhall happen all the $p$ firſt trials, and fail all the reſt, will therefore *(by Cor. to Prob.* I.) be $\frac{a^p}{\overline{a+b}|^p} \times \frac{b^{n-p}}{\overline{a+b}|^{n-p}} = \frac{a^p b^{n-p}}{\overline{a+b}|^n}$: but as there is the ſame probability for its happening any other $p$ aſſigned trials, and failing the reſt, it is manifeſt that as many different ways as $p$ things can be taken or combined in $n$ things, juſt ſo often ought the ſaid probability $\frac{a^p b^{n-p}}{\overline{a+b}|^n}$ be repeated to give the value ſought; wherefore that number of ways or combinations being $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$, &c. to $p$ factors, *(by Prob.* III.) $\frac{a^p b^{n-p}}{\overline{a+b}|^n} \times \frac{n}{1} \times \frac{n-2}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c. will conſequently be the value. Q. E. I.

## COROLLARY.

Hence, if $p$ be taken equal 0, 1, 2, 3, &c. ſucceſſively, the ſaid value will become $\frac{b^n}{\overline{a+b}|^n}$, $\frac{nb^{n-1}a}{\overline{a+b}|^n}$, $n \times \frac{n-1}{2} \times \frac{b^{n-2}a^2}{\overline{a+b}|^n}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^{n-3}a^3}{\overline{a+b}|^n}$, for the probability of happening *preciſely*, 0, 1, 2, 3, &c. times reſpectively; where, the ſeveral terms are thoſe of the binomial $b+a$ raiſed to the power whoſe index is $n$, divided by that power. And therefore if from the binomial $b+a$ raiſed to the $n$ power, all the terms where the indices of $a$ are leſs than $p$, be taken and divided by $\overline{a+b}|^n$, the quotient will, it is manifeſt, expreſs the probability that the propoſed event ſhall not happen ſo often as $p$ times in the given number $n$ of trials: but if the remaining terms, or thoſe where the indices of $a$ are not leſs that $p$, be divided by $\overline{a+b}|^n$, the quotient will then be the probability that it ſhall happen, at leaſt, $p$ times in thoſe trials.

EXAMPLE

## EXAMPLE I.

*A*, with four dice, undertakes to throw one ace and no more at the firſt throw; what is the odds againſt him?

By conſidering the dice as thrown one by one, and the four throws as ſo many trials, we have $n=4$, $p=1$, $a=1$, $b=5$, and therefore $\frac{nb^{n-1}a}{\overline{a+b}|^{n}}$ equal to $\frac{1\times5^{3}}{6^{4}}\times4=\frac{125}{324}$ for the probability of throwing one ace preciſely; wherefore that of the contrary is $\frac{199}{324}$, and the required odds 199 to 125, or 8 to 5 nearly.

## EXAMPLE II.

Suppoſing one to throw up nine halfpence; what is the odds that there come up more than three heads?

Here (by ſuppoſing as in the above caſe) $n$ will be $=9$, $a=1$, $b=1$; wherefore taking the 4 firſt terms, $1+9+9\times\frac{8}{2}+9\times\frac{8}{2}\times\frac{7}{3}$ $(=130)$ of $1+1$ raiſed to the 9th power, and dividing their ſum by $\overline{1+1}|^{9}$ $(=512)$ according to the corollary, there will be $\frac{65}{256}$ for the probability that there come not up four heads; therefore the odds are exactly as 191 to 65.

## EXAMPLE III.

There is a lottery conſiſting of a great number of tickets, whereof the prizes are to the blanks in the given proportion of 1 to 3: what is the odds that in taking ſeven tickets there ſhall come out two prizes?

Suppoſe the tickets to be taken one by one; and let it be firſt inquired what the probability is that there come not out two prizes; then $n$ being $=7$, $a=1$, $b=3$, and $p=2$, we ſhall have $\frac{b^n+nb^{n-1}a}{\overline{a+b}|^n}=\frac{3645}{8192}$ for the anſwer; and therefore the required odds will be as 4547 to 3645. But it muſt be obſerved that this proportion is not exactly true, as in the forgoing caſes; for here the ratio of $a$ to $b$, or the probability of drawing a prize, will not every trial be exactly the ſame, but greater or leſſer according as a greater or leſſer number of blanks has been before taken; whereas in dice, &c. the probability of throwing a given face, &c. ever continues the ſame. However when the number of tickets is large, the alteration of the ratio $\frac{a}{b}$ being then ſmall, this method of ſolution will be ſufficiently near the truth; but in other caſes, or where a rigid accuracy is inſiſted on, it will be beſt to have reference to the 8th problem.

N. B. *That an expreſſion placed in a parentheſis, immediately after any product or ſeries, is every where put to ſhew the number of factors, or terms, to which that product or ſeries is to be continued; as (p) at the end of* $n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}$, *or* $n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\ (p)$ *ſhews the product* $n\times\overline{n-1}\times\overline{n-2}$, *&c. is to be continued to as many factors as there are units in* (*p*).

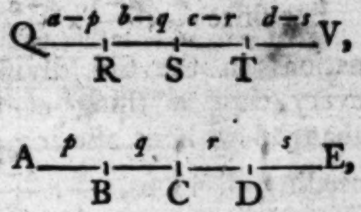
## PROBLEM VI.

*There is a given Number of each of ſeveral Sorts of Things (of the ſame Shape and Size); as* (*a*) *of the*

*the firſt Sort, ($b$) of the ſecond, &c. put promiſcuouſly together; out of which a given Number ($m$) is to be taken, as it happens: to find the Probability that there ſhall come out preciſely a given Number of each ſort, as ($p$) of the firſt, ($q$) of the ſecond, ($r$) of the third, &c.*

## SOLUTION.

In order to render the ſolution of this problem as eaſy as may be, let the things of the firſt ſort be ſelected from the reſt, and of thoſe conceive the given number $p$ to be placed between A and B, or in the firſt diviſion of the line AE, and the reſt between Q and R, or in the firſt diviſion of QV: in like manner, let the things of the ſecond and third ſorts be ſelected and diſpoſed of after the ſame method in the ſecond and third diviſions of thoſe lines reſpectively: and let us firſt inquire how many combinations may be made of $m$ things in the line AE, by mutually changing, one by one, the things in that line for thoſe in the other line QV, under this reſtriction, that in every combination the number of things of each ſort ſhall ſtill continue the ſame: then becauſe the ſame number of things of each ſort, or in each diviſion of both lines, is to be preſerved, a thing of the firſt, ſecond, or third diviſions of the lower muſt always be changed for one in the correſponding diviſion of the upper; and therefore



fore it is manifeſt, that as many ways as $p$ things can be taken in $a$ things, which is $\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}$ ($p$) *Prob.* 3, ſo many combinations may be made in the line AE, or QV, without changing any one of the things beſides thoſe in the firſt diviſions of thoſe lines: and for the ſame reaſons it is evident, that the number of combinations in each of the ſame lines, without changing any thing beſides thoſe in the ſecond or third, &c. diviſions, is $\frac{b}{1}\times\frac{b-1}{2}\times\frac{b-2}{3}$, ($q$) or $\frac{c}{1}\times\frac{c-1}{2}\times\frac{c-2}{3}$ ($r$), &c. reſpectively.

But ſince $\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}$ ($p$), the ſaid combinations in the firſt diviſion, may be repeated every time a thing of the ſecond diviſion is changed without altering any thing in the ſucceeding diviſions, $\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}$ ($p$) into $\frac{b}{1}\times\frac{b-1}{2}\times\frac{b-2}{3}$ ($q$) muſt conſequently be the number of all the combinations that can be produced by changing the things of the firſt and ſecond, without affecting thoſe of the other diviſions. But this number of combinations may in like manner be repeated every time a thing of the third diviſion is changed; and therefore all the combinations that can be produced under the propoſed reſtriction, without changing any of the things beſides thoſe of the three firſt diviſions, will be $\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}$ ($p$) in. $\frac{b}{1}\times\frac{b-1}{2}\times\frac{b-2}{3}$ ($q$) in. $\frac{c}{1}\times\frac{c-1}{2}\times\frac{c-2}{3}$ ($r$); from whence the proceſs and law of continuation are manifeſt. Wherefore, having now found all the poſſible ways that the things can be taken to have the propoſed numbers of each ſort, we are next to ſee how many ways the ſame number ($m$) of things may be had without any reſtriction. This number of ways, if $n$ be put $= a + b + c$, &c. the whole number of things will be $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$ ($m$) (*by the aforeſaid Problem*): and therefore

$$\frac{\frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3} (p) \text{ in. } \frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3} (q) \text{ in. } \frac{c}{1} \times \frac{c-1}{2} \times \frac{c-2}{3} (r)}{\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} (m)}$$

muſt conſequently be the value. Q. E. I.

## COROLLARY I.

Hence if the number of things of each ſort be equal, and thoſe propoſed to be taken of each alſo equal, and $w$ be put for the number of ſorts, the probability, it is manifeſt, will become

$$\frac{\left.\overline{\frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3} (p)}\right|^{w}}{\left.\overline{\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} (m)}\right|}.$$

## COROLLARY II.

When there are only two ſorts of things, then $c, d, r, s,$ &c. become $=0$, $a+b=n$, $p+q=m$, and

$$\frac{\frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3} (p) \text{ in. } \frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3} (q)}{\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} (m)}$$

equal to the required value in that caſe; which, when it is propoſed that no one of the firſt of the two ſorts ſhall be taken, will become $\frac{b}{n} \times \frac{b-1}{n-1} \times \frac{b-2}{n-2} \times \frac{b-3}{n-3}$ ($m$), becauſe then $p$ is $=0$, and $q=m$.

## EXAMPLE I.

In a lottery conſiſting of 99 tickets, whereof there are two prizes of 1000*l.* and five of 100*l.* What is the probability that in taking 6 tickets there ſhall be juſt one prize of each of thoſe two ſorts? Here $n$ being $=99$, $m=6$, $a=2$, $b=5$, $c=92$, $d=0$, &c. $p=1$, $q=1$, $r=4$, $s=0$, &c. the Value ſought, by the general theorem, will be equal to $\frac{\frac{2}{1} \times \frac{5}{1} \times \frac{92}{1} \times \frac{91}{2} \times \frac{90}{3} \times \frac{89}{4}}{\frac{99}{1} \times \frac{98}{2} \times \frac{97}{3} \times \frac{96}{4} \times \frac{95}{5} \times \frac{94}{6}}$ $= \frac{23 \times 25 \times 89 \times 91}{19 \times 22 \times 49 \times 94 \times 97} = \frac{4656925}{186760876}$, or $\frac{1}{40}$ nearly.

## EXAMPLE II.

Suppoſe one to draw four cards out of the whole pack; and let it be required to find the odds that there ſhall not come out juſt one of each ſort, as one heart, one diamond, &c. Becauſe, $n$, the whole number of things (or cards) is here $=52$, thoſe taken (or $m$) $=4$; ($w$) the number of ſorts $=4$; $a$, $b$, $c$ and $d$, each $=13$; and $p$, $q$, $r$ and $s$ each $=1$: By Cor. I. we have $\frac{\overline{13}\rceil^{4}}{\frac{52}{1} \times \frac{51}{2} \times \frac{50}{3} \times \frac{49}{4}} = \frac{2197}{20825}$ for the probability of happening; therefore that of the contrary is $\frac{18628}{20825}$, and the required odds as 18628 to 2197, or nearly as 8 to 1.

EXAMPLE

## EXAMPLE III.

Let there be a heap of twenty cards, wherein are ſeven diamonds, ſix hearts, four ſpades, and three clubs; to find the probability that in drawing eight of them, at a venture, there ſhall come out juſt three diamonds and two hearts.

Here, becauſe the anſwer is reſtrained only to the taking of three diamonds, two hearts, and three other cards that are neither diamonds nor hearts, theſe laſt, or all the black ones, in reſpect to the general theorem, may, it is manifeſt, be conſidered as one ſort. And we ſhall have, $a=7$, $b=6$, $c=7$, $n=20$, $m=8$, $p=3$, $q=2$, $r=3$; and therefore $\dfrac{\frac{7}{1}\times\frac{6}{2}\times\frac{5}{3}\times\frac{6}{1}\times\frac{5}{2}\times\frac{7}{1}\times\frac{6}{2}\times\frac{5}{3}}{\frac{20}{1}\times\frac{19}{2}\times\frac{18}{3}\times\frac{17}{4}\times\frac{16}{5}\times\frac{15}{6}\times\frac{14}{7}\times\frac{13}{8}}$ $=\dfrac{35\times35}{17\times19\times26}=\dfrac{1225}{3978}$ the probability: wherefore the odds is as 2753 to 1225, or as 9 to 4 nearly.

## EXAMPLE IV.

There is a lottery conſiſting of 10,000 tickets, among which are three particular prizes; what is the odds that a perſon in taking 2000 of them ſhall not have all thoſe prizes? As it it matters not, here, whether there be other prizes in the lottery beſides theſe three, the other 9997 tickets may be all conſidered as blanks; and then the probability of taking the three prizes with 1997 of theſe blanks, (*by the theorem in Cor.* II.) is $=\dfrac{\frac{9997}{1}\times\frac{9996}{2}\times\frac{9995}{3}\text{ \&c. to 1997 factors}}{\frac{1000}{1}\times\frac{9999}{2}\times\frac{9998}{3}\text{ \&c. to 2000 factors}}$: but as this expreſſion, by reaſon of the great multitude of factors it involves, muſt be impracticable without a proper method of reduction, and as this will always be the caſe where the number of things taken is very large, it may not be improper to ſhew here how the theorem itſelf may in thoſe caſes be contracted. In order thereto, equally multiplying both numerator and denominator by $1\times2\times3\times4\ (m)$, our ſaid theorem *(Vid. Cor.* II.) will, it is manifeſt, firſt become

$$\frac{\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}\,(p)\ \text{into}\ b\times\overline{b-1}\times\overline{b-2}\,(b)\ \text{into}\ m\times\overline{m-1}\times\overline{m-2}\,(p)}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\overline{n-5}\,(m)};$$

where, by breaking the denominator into two parts, ſo that the firſt factor of the latter may be $=b$, it will next ſtand thus,

$$\frac{\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}\,(p)\ \text{in.}\,b\times\overline{b-1}\times\overline{b-2}\,(q)\ \text{in.}\,m\times\overline{m-1}\times\overline{m-2}\,(p)}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\,(a)\ \text{into}\ b\times\overline{b-1}\times\overline{b-2}\,(m-a)};$$

where equally dividing by $b\times\overline{b-1}\times\overline{b-2}\ (m-a)$ it becomes

$$\frac{\frac{a}{1}\times\frac{a-1}{2}\,(p)\ \text{in.}\,\overline{a+b-m}\times\overline{a+b-m-1}\,(a+q-m)\ \text{in.}\,m\times\overline{m-1}\,(p)}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\,(a)}$$

where putting $k$ for $a+b-m=$ the things remaining after $(m)$ the propoſed number is taken, and for $a+q-m$ its equal $a-p$, it will, at length, be reduced to

$$\frac{\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}\,(p)\ \text{in.}\,k\times\overline{k-1}\times\overline{k-2}\,(ap)\ \text{in.}\,m\times\overline{m-1}\times\overline{m-2}\,(p)}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\overline{n-5}\times\overline{n-6}\,(a)}$$

equal to the probability of taking preciſely $p$ things of that ſort whoſe number is $a$: by help of which it will now be eaſy to proceed in the proſecution of our example; for here $n$ being $=10000$, $m=2000$, $k=8000$, $a=3$, and $p=3$, the ſaid expreſſion becomes

$$\frac{\frac{3}{1}\times\frac{2}{2}\times\frac{1}{3}\times 2000\times 1999\times 1998}{10000\times 9999\times 9998}=\frac{1997001}{249925005}$$

equal to the probability of taking all the three prizes; wherefore the required odds is as 247928004 to 1997001, or as 124 to 1 nearly.

## EXAMPLE V.

The ſame things being ſuppoſed as in the laſt caſe; let it be required to find the exact odds that there comes out one or more prizes?

In this caſe, it is manifeſt, the anſwer may be obtained by ſeeking ſeverally the probabilities of taking one, two, and three prizes, and adding them together, &c. But by firſt making the probability of taking no prize at all the ſubject of our enquiry, it will be eaſily had at one operation: for then $p$ being $=0$, the general expreſſion (contracted as above) becomes $\frac{8000}{10000}\times\frac{7999}{9999}\times\frac{7998}{9998}=\frac{127952004}{249925005}$; and therefore the odds are as 127952004 to 121973001, that there comes out no prize.

## PROBLEM VII.

*Suppoſing a great, but given Number of each of two Sorts of Things to be put promiſcuouſly together; to find how many muſt be taken out of the Whole, to make it an equal Chance that they ſhall all come out of one given Sort.*

## SOLUTION.

If $n$ be put for the whole number of things, $b$ thoſe of the given ſort, and $m$ the number required; it it manifeſt from Cor. II. to the laſt problem, that the probability of taking all the $m$ things of that ſort will be $\frac{b}{n} \times \frac{b-1}{n-1} \times \frac{b-2}{n-2}\,(m)$; which, by the queſtion, muſt here be $=\frac{1}{2}$; but $\frac{b}{n}$, $\frac{b-1}{n-1}$, $\frac{b-2}{n-2}$ being nearly equal; becauſe $b$ and $n$ are large numbers, $\frac{b}{n} \times \frac{b-1}{n-1} \times \frac{b-2}{n-2}\,(m)$ $(=\frac{1}{2})$ will be $\frac{b}{n} \times \frac{b}{n} \times \frac{b}{n}\,(m)$ or $\overline{\frac{b}{n}}\big|^{m}$ very nearly; whence in logarithms $m$ into Log. $b -$ log. $n = -$ log. 2, and $m = \frac{\text{log. } 2.}{\text{log. } n - \text{log. } b}$. Q. E. I.

### EXAMPLE.

In a lottery conſiſting of 100,000 tickets, in which the blanks are to the prizes as 9 to 1, how many ought one to take, to make it an equal chance that there ſhall come out one or more prizes? Or, which is the ſame, that there ſhall be no one of them prizes? Here $n$ being 100,000, and $b$ 90,000, the general expreſſion becomes $\frac{.30103}{5 - 4.95424} = 6.58$; which not coming out a whole number, ſhews there can be no exact equality of chance in this caſe, 6 being too ſmall, and 7 too great a number.

## PROBLEM VIII.

*The ſame being ſuppoſed as in the laſt Problem; to find the Probability that in taking a given Number (m) of thoſe Things, there ſhall not come out a given Number (p) of one Sort.*

### SOLUTION.

## SOLUTION.

This problem may be ſolved in any caſe, from Cor. II. to Prob. VI. by finding at ſeveral operations the probability of taking preciſely all the inferior numbers to $p$ the propoſed one. But theſe operations may be very much contracted; for if (as before) $a$ be put for the things of one ſort, $b$ thoſe of the other, and $n$ the whole number of both ſorts, it is manifeſt from thence that

$$\frac{\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}\ (p)\ \text{into}\ \frac{b}{1}\times\frac{b-1}{2}\times\frac{b-2}{3}\ (m-p)}{\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\ (m)}$$

will be the probability of taking $p$ things preciſely of that ſort whoſe number is $a$. Therefore if $p$ be ſuppoſed equal to 0, 1, 2, 3, &c. ſucceſſively, and A be put $=\frac{b}{n}\times\frac{b-1}{n-1}\times\frac{b-2}{n-2}\ (m)=$ the probability of taking none of the ſaid ſort, and $b-m+1=b$; then will A, $\frac{m}{b}\times\frac{a}{1}A$, $\frac{m}{b}\times\frac{m+1}{b+1}\times\frac{a}{1}\times\frac{a-1}{2}\ A$, $\frac{m}{b}\times\frac{m-1}{b+1}\times\frac{m-2}{b+2}\times\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}\ A$, &c. be the reſpective probabilities of drawing 0, 1, 2, 3, &c. of that ſort, preciſely: wherefore $A+\frac{m}{b}\times\frac{a}{1}A+\frac{m-1}{b+1}\times\frac{a-1}{2}\times B+\frac{m-2}{b+2}\times\frac{a-2}{2}C+\frac{m-3}{b+3}\times\frac{a-3}{3}D$, &c. continued to as as many terms as there are units in $p$, will be the probability ſought; B, C, D, &c. denoting the preceding terms.

EXAMPLE.

## EXAMPLE.

In a heap of 10 cards whereof one half are red and the other half black; what is the probability that in drawing 5 of them at a venture, there ſhall come out 3 red ones? or, which is the ſame, that there ſhall not come out 3 black ones? Here $a$ being $=5$, $b=5$, $n=10$, $p=3$, and $m=5$; A or $\frac{b}{n}\times\frac{b-1}{n-1}$ $(m)$ will be $\frac{1}{252}$, and and $h$ $(=b-m+1)=1$; wherefore $A+\frac{m}{b}\times\frac{a}{1}A+\frac{m-1}{b+1}\times\frac{a-1}{2}B$ $(p)$ becomes $\frac{1}{252}+\frac{25}{252}+\frac{100}{252}=\frac{1}{2}$; which therefore is the required value in this caſe.

## PROBLEM IX.

*The ſame being ſtill ſuppoſed as in the preceding Problems, and that the ſaid Things are to be taken one by one as it happens; to find the Probability that the (p) firſt ſhall all come out of the firſt Sort, and the next after of the contrary Sort.*

## SOLUTION.

The number of the things of the firſt ſort being $a$, that of the ſecond ſort $b$, and the whole number of both ſorts $n$, the probability of taking one of the firſt ſort firſt, will, it is manifeſt, be $\frac{a}{n}$; If this ſhould happen, that is, if one of that ſort ſhould be actually taken, then there remaining only $a-1$ of that ſort, the probability of taking of thoſe next would be $\frac{a-1}{n-1}$; wherefore the probability of taking both the two firſt of the ſaid ſort will be $\frac{a}{n}\times\frac{a-1}{n-1}$ (*Propoſition* I.) From the manner of the proceſs it is evident that the probability of taking all the firſt $p$ of this ſort will be $\frac{a}{1}\times\frac{a-1}{n-1}\times\frac{a-2}{n-2}\times\frac{a-3}{n-3}$ $(p)$: but if theſe ſhould be ſo taken, the probability of taking one of the other ſort next will be $\frac{b}{n-p}$; wherefore $\frac{a}{n}\times\frac{a-1}{n-1}\times\frac{a-2}{n-2}$ $(p)$ into $\frac{b}{n-p}=\frac{b}{n}\times\frac{a}{n-1}\times\frac{a-1}{n-2}\times\frac{a-2}{n-3}\times\frac{a-3}{n-4}$ $(p+1)$ is the value that was to be found.

## PROBLEM X.

*To determine the Odds at Bowls, or other Games of the like Nature, in any Circumſtance of the Play.*

## SOLUTION.

Firſt, ſuppoſe the Players to be only *A* and *B*, or, that there are only 2 bowls of a ſide; then, as the probability that either of them ſhall win juſt 1 bowl at an end is the ſame as that of taking 2 things out of 4 things of 2 ſorts, ſo that the firſt may come out of the firſt ſort, and the other of the contrary; the ſaid probability, it is manifeſt, will be $\frac{2}{4}\times\frac{2}{3}=\frac{1}{3}$, by the laſt problem; and, for the like reaſon, the probability of winning 2 bowls at an end will, by the ſame, be had $=\frac{2}{4}\times\frac{2}{3}\times\frac{1}{2}=\frac{1}{6}$. And the like may be had in any other caſe when the players are a greater number.

This being premiſed, *let* A *want* 2 *of being up*, B 1; *and the value of the thing played for be denoted by* 1.

If *A* gets juſt 1 at the firſt end, of which the probability is $\frac{1}{3}$ *(per above)* he will be in the ſame circumſtance with his antagoniſt, and therefore entitled to $\frac{1}{2}$ the whole ſtake, or, $\frac{1}{2}$; wherefore his expectation on that event, while it remains uncertain, will be $\frac{1}{3}$ of $\frac{1}{2}=\frac{1}{6}$. But if he ſhould get 2 he will be entitled to (1) the whole ſtake; wherefore as the probability of getting 2 is $\frac{1}{6}$; his expectation on this event will be, alſo, $\frac{1}{6}$, which added to the foregoing, will conſequently give ($\frac{1}{3}$) the total expectation of *A* in the propoſed circumſtance; which ſubſtracted from (1) the whole ſtake, leaves that of $B=\frac{2}{3}$, and therefore the odds are as 2 to 1 in this caſe.

*Let* A *want* 3, *and* B *as before*. Becauſe the probability that *A* gets juſt one at the firſt end, or there brings the play to the circumſtance of the above caſe, is $\frac{1}{3}$; and as his expectation in that circumſtance will alſo be $\frac{1}{3}$ *(per above)* his preſent expectation on that event will therefore be $\frac{1}{9}$. But if he ſhould get 2, he will be entitled to half the whole ſtake, or $\frac{1}{2}$; which therefore multiplied by $\frac{1}{6}$, the probability of getting 2, gives $\frac{1}{12}$ for his expectation on this event; whence $\frac{1}{12}+\frac{1}{9}=\frac{7}{36}$, the ſum of theſe 2, muſt be the total expectation of *A* in this caſe; hence that of $B=\frac{26}{36}$, and the odds as 29 to 7.

*Let* A *want* 3, *and* B 2. Now *A* muſt either get 1 or 2 at the firſt end; or, *B* 1 or 2 at the ſame end.

If *A* gets 1 he is entitled to $\frac{1}{2}$ the ſtake; which multiplied by $\frac{1}{3}$, the probability of it, is $\frac{1}{6}$, the expectation on that event.

If he gets 2 he is entitled to $\frac{2}{3}$, *(by the firſt caſe)*; therefore the expectation on this event is $\frac{1}{6}\times\frac{2}{3}=\frac{1}{9}$.

If *B* gets 1, *A* will be entitled to $\frac{7}{18}$, *(by the laſt caſe)* and therefore on this event his expectation is $\frac{1}{6}\times\frac{7}{18}=\frac{7}{108}$. But if *B* gets 2, the expectation of *A* will be nothing at all; therefore $\frac{1}{6}+\frac{1}{9}+\frac{7}{108}=\frac{37}{108}$, the ſum of thoſe 3, is the total expectation of *A* in this cafe. And by proceeding on in this manner, the expectations of the gameſters may be determined in any other circumſtance of the play; and tables calculated to exhibit the odds, not only when the players are 2, but a greater number. The firſt of the 2 following tables ſhews the odds when there are only 2 players; and the laſt, the limits of the leaſt odds when the players are indefinite, the odds in any circumſtance of the play growing leſs as the number of players is encreaſed.

| | | Truly. | Nearly. |
|---|---|---|---|
| 2 | 1 | 2 : 1 | 2 : 1 |
| 3 | | 29 : 7 | 4 $\frac{1}{7}$ : 1 |
| 4 | | 95 : 13 | 7 $\frac{1}{3}$ : 1 |
| 5 | | 601 : 47 | 12 $\frac{4}{5}$ : 1 |
| 3 | 2 | 71 : 37 | 1 $\frac{11}{12}$ : 1 |
| 4 | | 247 : 77 | 3 $\frac{1}{5}$ : 1 |
| 5 | | 272 : 52 | 5 $\frac{1}{5}$ : 1 |
| 4 | 3 | 607 : 365 | 1 $\frac{2}{3}$ : 1 |
| 5 | | 8507 : 2157 | 2 $\frac{2}{3}$ : 1 |
| 5 | 4 | 21485 : 13507 | 1 $\frac{1}{2}$ : 1 |

| | | Truly. | Nearly. |
|---|---|---|---|
| 2 | 1 | 5 : 3 | 1 $\frac{2}{3}$ : 1 |
| 3 | | 23 : 9 | 2 $\frac{5}{9}$ : 1 |
| 4 | | 101 : 27 | 3 $\frac{3}{4}$ : 1 |
| 5 | | 431 : 81 | 5 $\frac{1}{3}$ : 1 |
| 3 | 2 | 77 : 51 | 1 $\frac{1}{2}$ : 1 |
| 4 | | 175 : 81 | 2 $\frac{1}{6}$ : 1 |
| 5 | | 1535 : 513 | 3 : 1 |
| 4 | 3 | 601 : 423 | 1 $\frac{3}{7}$ : 1 |
| 5 | | 5311 : 3781 | 1 $\frac{2}{10}$ : 1 |
| 5 | 4 | 18909 : 13859 | 1 $\frac{1}{3}$ . 1 |

Now

Now by the help of theſe tables the odds may be nearly had in any intermediate caſe as follows, *viz.* In the propoſed circumſtance, be it what it will, from the ſaid tables, find the probabilities of winning in thoſe two caſes for which they are calculated, and let thoſe probabilities be denoted by $p$ and $P$ reſpectively, and let $n$ be the number of players in the caſe propoſed; then will $P-\frac{P-p}{\frac{1}{2}n}$ be the probability of winning in that caſe, nearly.

## EXAMPLE.

Suppoſe the number of players to be 6; and one ſide to want 2 of being up, and the other 4. Then by the tables the odds will be as $3\frac{1}{5}$ to 1, and $2\frac{1}{6}$ to 1; and therefore the probabilities of winning $\frac{5}{21}$, and $\frac{6}{19}$ reſpectively, (in thoſe caſes for which the tables are calculated under the ſame circumſtance); wherefore $P$ is here $=\frac{6}{19}$, and $p=\frac{5}{21}$, and therefore $P-\frac{P-p}{\frac{1}{2}n}=\frac{347}{1197}=\frac{2}{7}$ nearly = the probability of winning in the caſe propoſed; whence the odds are as 5 to 2 nearly.

## PROBLEM XI.

*Two Perſons (or Parties)* A *and* B, *play together with (n) bowls a Side; the Skill of* A *to that of* B, *or the Odds that any aſſigned Bowl of* A *is nearer to the Jack than any aſſigned Bowl of* B, *is in the given Ratio of (a) to (b); to find the Probability that* A *has of getting a given Number (p) or more, and alſo that of getting that Number preciſely at any End aſſigned.*

SOLUTION.

## SOLUTION.

Since the odds, or chances, that any aſſigned bowl of *A* comes nearer the jack than any aſſigned bowl of *B* are as $a$ to $b$; each bowl of *A* may be ſuppoſed to contain $a$ chances, and each one of *B*, $b$ chances; and then the number of chances in all the bowls of *A* being $na$, and the total number of all the chances $na+nb$, the probability that ſome one of *A*'s bowls comes nearer to the jack than any one of *B*'s will, it is manifeſt, be $\frac{na}{na+nb}$, or, if $ar$ be put $=n\times\overline{a+b}$, equal to $\frac{n}{r}$. Now if 1 of *A*'s bowls ſhould come neareſt them, he having then $\overline{n-1}$ bowls, or $a\times\overline{n-1}$ chances remaining, the probability of his having a ſecond bowl nearer the jack than any one of *B*'s would, it is manifeſt, be $\frac{a\times\overline{n-1}}{an+bn-a}$ or $\frac{a\times\overline{n-1}}{ra-a}$ $=\frac{n-1}{r-1}$; wherefore the probability that *A* ſhall have 2 bowls nearer to the jack than any one of *B*'s is $\frac{n}{r}\times\frac{n-1}{r-1}$ *(by Cor. to Prob.* I.) In like manner, the probability that *A* ſhall have $p$ bowls nearer to the jack than any one of *B*'s, will be found $\frac{n}{r}\times\frac{n-1}{r-1}\times\frac{n-2}{r-2}\times\frac{n-3}{r-3}\,(p)$; which is an anſwer to the firſt part of the queſtion: now if that ſhould happen, of which this laſt expreſſion is the probability, then there remaining $\overline{r-p}\times a$ chances, and $nb$ of them in favour of *B*, the probability that one of his bowls comes next will be $\frac{nb}{\overline{r-p}\times a}$; therefore (*by* *the ſame Cor.)* $\frac{nb}{\overline{r-p} \times a}$ into $\frac{n}{r} \times \frac{n-1}{r-1} \times \frac{n-2}{n-2} \times \frac{n-3}{r-3}$ $(p) = \frac{nb}{ra} \times \frac{n}{r-1} \times \frac{n-1}{r-2} \times \frac{n-2}{r-3}$ $(p+1)$ will be the value. Q. E. I.

Note, *By help of this Theorem, and the method of proceeding in the foregoing problem, the probability, or odds of winning the game, may be determined in any caſe whatever, where the ratio of* a *to* b, *or the proportion of ſkill, can be aſcertained.*

## PROBLEM XII.

A *and* B *playing together with ſingle Bowls, Coits, or Pieces, &c. the former finds by Experience that he can upon an Equality of Chance, undertake to win (n) Times, before his Antagoniſt once: what is the Proportion of Skill of the two Gameſters, or their Chances of winning at any aſſigned Trial.*

## SOLUTION.

Let the required ratio be that of $a$ to $b$, or, which is to the ſame effect, let $\frac{a}{a+b}$ be the probability of *A*'s winning at the firſt trial; then by Cor. to Prob. I. $\frac{a^n}{\overline{a+b}|^n}$ will be the probability of his winning all the $n$ firſt times, or trials, which ∵ by the queſtion muſt be $= \frac{1}{2}$; whence $2a^n = \overline{a+b}|^n$ and $a \times \overline{2^{\frac{1}{n}} - 1} = b$; therefore, as $1 : 2^{\frac{1}{n}} - 1 :: a : b$. Q. E. I.

PROBLEM

## PROBLEM XIII.

A *and* B, *whoſe Proportion of Skill, or Chances for winning any aſſigned Game, are as (a) to (b), play together; the former wants (p) Games of the whole Set, and the latter (q); what are their reſpectively Probabilities of winning?*

### SOLUTION.

Suppoſe the play to be continued after the ſet is out till ſuch time as $p+q-1$ games are expired; and a ſpectator *E* to wager with *F* that *A* beats his adverſary *p* of theſe games. Then, *(by Cor. to Prob.* V.) it is manifeſt the probability of *E*'s winning will be

$$\frac{a^n + na^{n-1}b + n \times \frac{n-1}{2} a^{n-2}b^2 \ldots\ldots n \times \frac{n-1}{2} \&c.\, a^p b^{n-p}}{\overline{a+b}|^n};$$

but this is the probability that *A* wins the ſet; becauſe if he gets *p* of thoſe $p+q-1$ games he muſt loſe fewer than *q* of them, and therefore get *p* before he loſes *q*; and becauſe it is evident that whatever the chance of *E* may be in reſpect of winning and loſing, that of *A* muſt be the ſame, and *vice verſa.* And therefore, for the ſame reaſon, the probability that *B* wins the ſet muſt be

$$\frac{b^n + nb^{n-1}a + n \times \frac{n-1}{2} b^{n-2}a^2}{\overline{a+b}|^n} \ldots\ldots$$

$$\frac{n \times \frac{n-1}{2} \&c.\, b^q a^{n-q}}{\overline{a+b}|^n}$$ Q. E. I.

EXAMPLE.

## EXAMPLE.

Suppoſe *A* to want 5 of being up, *B* 3, and the ſkill of the 2 gameſters to be equal: then will $p=5$, $q=3$, $a=1$, $b=1$, $n=7$, and $\frac{1+7+21+35+35}{2^7}$ $(=\frac{99}{128})$ and $\frac{21+7+1}{2^7}$ $(=\frac{29}{128})$, be the probabilities of winning; therefore the odds that *B* beats *A* are as 99 to 29.

## PROBLEM XIV.

*A given Number of Gameſters,* A, B, C, *&c. whoſe Chances for winning any aſſigned Game are in the given Ratio of a, b, c, &c. play together;* A *wants p Games of the whole Set,* B, *q, and* C, *r, &c. What are their ſeveral Probabilities of winning?*

## SOLUTION.

Raiſe $a+b+c$, &c. to the power whoſe index is equal to the leaſt of the given numbers $p$, $q$, $r$, &c. or, if there be no leaſt, to one of the leaſt equal ones, as $p$; and from that power take out the term wherein the exponent of the correſponding quantity $a$ is equal to $p$; and if there be any terms wherein the indices of $b$, $c$, &c. are equal to $q$, $r$, &c. take thoſe terms alſo, and having divided each of the terms ſo taken by $\overline{a+b+c,\ \&c.}|^p$, or $b^p$, which is ſuppoſed equal to it, place the ſeveral quotients each in different columns, marked (for diſtinction ſake) *A*, *B*, and *C*, ſo that the ſaid power of $a$ may be in the column *A*, that of $b$ (if any) in the column *B*, &c. And then, having proceeded thus far, multiply the remaining terms of the ſaid power by $a+b+c$, &c. and from the product ſelect all the terms where the indices of the powers of $a$, $b$, $c$, &c. are reſpectively equal to $p$, $q$, $r$, &c. And having divided each by $b^{p+1}$, place them according to the foregoing method, in the columns *A*, *B*, *C*, &c. Let the laſt remainder be multiplied in like manner by $a+b+c$, &c. and ſelect, again, out of this product the terms wherein the exponents of $a$, $b$, $c$, &c. are reſpectively equal to $p$, $q$, $r$, &c. and having divided each by $b^{p+2}$, let the quotients be diſpoſed of as before. And proceed on in this ſame manner, repeating the operation till all the terms are exhauſted; then the quantities that are, at laſt, found in the columns *A*, *B*, *C*, &c. will be, reſpectively, the required probabilities of winning.

## EXAMPLE.

Suppoſe the number of players to be 3, as *A*, *B* and *C*, and they want 1, 2, and 3 reſpectively.

Having firſt raiſed $a+b+c$ to the 1 ($p$) power, and prepared 3 columns *A*, *B*, *C*, I take $a\ (=a^p)$ from the ſaid power, divide it by $b\ (=\overline{a+b+c}^p)$ and place the quotient $\frac{a}{b}$ in the column *A*; then multiply the remainder $b+c$ by

$$\begin{array}{l|l}
\hline
A & \frac{a}{b}+\frac{ab+ac}{bb}+\frac{2abc+acc}{b^3}+\frac{3abc^2}{b^4} \\
\hline
B & \frac{bb}{bb}+\frac{2bbc}{b^3}+\frac{3bbcc}{b^4} \\
\hline
C & \frac{c^3}{b^3}+\frac{3bc^2}{b^4} \\
\hline
\end{array}$$

$a+b+c$, and from the product $ab+ac+bb+2bc+cc$, take the terms $ab+ac$, $bb$, divide each by $bb$, and place the 2 quotients $\frac{ab+ac}{bb}$, $\frac{bb}{bb}$ in the columns $A$ and $B$; multiply the laſt remainder $2bc+cc$, by $a+b+c$; from the product $2abc+acc+2bbc+3bcc+c^3$ ſelect $2abc+acc$, $2bbc$, and $c^3$, divide each of them by $b^3$, and diſpoſe of the quotients as before; laſtly, I multiply the new remainder $3bcc$ by $a+b+c$, divide each of the 3 terms in the product by $b^4$, and place the quotient in the columns $A$, $B$, and $C$ as above, and then find the total values in theſe 3 columns to be $\frac{a}{b}+\frac{ab+ac}{bb}+\frac{2abc+acc}{b^3}+\frac{3abcc}{b^4}$, $\frac{bb}{bb}+\frac{2bbc}{b^3}+\frac{3b^2c^2}{b^4}$, and $\frac{c^3}{b^3}+\frac{3bc^3}{b^4}$; theſe are reſpectively equal to the probabilities required; which, when $a$, $b$, and $c$ are equal, will therefore become $\frac{19}{27}$, $\frac{6}{27}$ and $\frac{2}{27}$.

Note, *The above method of ſolution is only a compounded caſe of Cor. to Prob. I. and therefore to ſuch as underſtand that well, the reaſons of this will not be difficult.*

## PROBLEM XV.

*There being* (*a*) *Chances for the happening of an Event, and* (*b*) *Chances for the Contrary at any aſſigned Trial; in how many Trials may one undertake that the ſaid Event ſhall happen* (*r*) *times?*

## SOLUTION.

Let $n$ be the number ſought; then

$$\frac{b^n+nb^{n-1}a+n\times\frac{n-1}{2}b^{n-2}a^2+n\times\frac{n-1}{2}\times\frac{n-2}{3}b^{n-3}a^3\,(r)}{\overline{a+b}|^n}$$

being the probability that event ſhall not happen $r$ times in $n$ trials (*by Prob.* V.) muſt therefore by the queſtion be $=\frac{1}{2}$; whence $b^n + nb^{n-1}a + n \times \frac{n-1}{2}b^{n-2}a^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3}b^{n-3}a^3\ (r) = \frac{\overline{a+b}|^n}{2}$; where ſubſtituting $pb$ for $a$ we have $1 + np + n \times \frac{n-1}{2}p^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3}p^3\ (r) = \frac{\overline{1+p}|^n}{2}$; from whence by the common methods of converging or infinite ſeries the value of $n$ may in any caſe be determined.

Suppoſe $a=b$, or $p=1$; then ſince $1+np+n \times \frac{n-1}{2}p^2$, $(r)$ the firſt half of the terms of the binomial $\overline{1+p}|^n$ expanded in a ſeries is equal to the reſt of the terms, or half the whole power, the whole number of terms muſt be $=2r$, and therefore $n$ the index of the power $=2r-1$, which is the required value in this caſe.

Let $\frac{a}{b}=p$ be very ſmall; then becauſe in this caſe $n$ muſt be very great, the numbers 1, 2, 3, &c. in the factors $n-1$, $n-2$, $n-3$, &c. may be rejected as inconſiderable in reſpect of $n$, which done, the equation becomes $1+pn+\frac{n^2p^2}{2}+\frac{n^3p^3}{2 \cdot 3}+\frac{n^4p^4}{2 \cdot 3 \cdot 4}\ (r) = \frac{\overline{1+p}|^n}{2}$; where, by ſubſtituting $r+v$ inſtead of $pn$, we get $1+r+v+\frac{\overline{r+v}|^2}{2}+\frac{\overline{r+v}|^3}{2 \times 3}+\frac{\overline{r+v}|^4}{2 \times 3 \times 4}\ (r) = \frac{\overline{1+p}|^{\frac{r+v}{p}}}{2}$ or, by bringing the ſeveral powers of $r+v$ into ſimple terms, it will be

$$\left|\begin{array}{l} 1,\ \text{in } 1+r+\frac{r^2}{2}\times\frac{r^3}{2.3}\ (r) \\ v\ \text{in } 1+r+\frac{r^2}{2}+\frac{r^3}{2.3}\ (r-1) \\ \frac{v^2}{2}\ \text{in } 1+r+\frac{r^2}{2}+\frac{r^3}{2.3}\ (r-2) \\ \frac{v^3}{2.3}\ \text{in } 1+r+\frac{r^2}{2}+\frac{r^3}{2.3}\ (r-3) \end{array}\right| = \frac{\overline{1+p}|^{\frac{r+v}{p}}}{2}$$

Put $e=1+r+\frac{r^2}{2}+\frac{r^3}{2.3}(r)$, $f=1+r+\frac{r^2}{2}+\frac{r^3}{2.3}(r-1)$ $g=1+r+\frac{r^2}{2}+\frac{r^2}{2.3}\ (r-2)$, &c. and $s=$ hyperb. log. of $2e$; then it will become $e+fv+\frac{gv^2}{2}+\frac{hv^3}{2.3}+\frac{jv^4}{2.3.4}+\frac{kv^5}{2.3.4.5}$ &c. $=\frac{\overline{1+p}|^{\frac{r+v}{p}}}{2}$, or in logarithms, $s+\frac{fv}{e}+\frac{ge-ff}{2ee}\times vv+\overline{\frac{h}{6e}-\frac{fg}{2ee}+\frac{f^3}{3e^3}}\times v^3$ &c. $=r+v$ in $\overline{1-\frac{p}{2}+\frac{p^2}{3}}$, &c. $=r+v$ very nearly, becauſe $p$ is ſuppoſed very ſmall; wherefore by reduction $v+\frac{ff-ge}{2e\times\overline{e-f}}\times v^2+\frac{3fg-eh+\frac{2f^3}{e}}{6e\times\overline{e-f}}\times v^3$, &c. $=e\times\frac{s-r}{e-f}$: whence by putting theſe known coefficients of the powers of $v$, equal to 1, $B$, $C$, &c. reſpectively, and $e\times\frac{s-r}{e-f}=z$, we have $v=z-Bz^2+\overline{2B^2-C}\times z^3+\overline{5BC-5B^3-D}\times z^4$ &c. and therefore $pn$ $(=r+v)$ $=r+z-Bz^2+\overline{2B^2-C}\times z^3$ &c. or, becauſe the ſeries converges very ſwift, $pn=r+z$ nearly; wherefore $\frac{r+z}{p}$, &c. is the required value in this caſe.

Hence if $r=1$, then will $e=1$, $f=0$, &c. $s=$ *hyp.* log. $2=69314$, $z=-.30686$, $pn=.69314$, and $n=\frac{.69314}{p}$.

If $r=2$, then $e=3$, $f=1$, $g=0$, &c. $s=$ *hyp.* log. $6=1.7917$, $z=-.31236$, and $pn=1.6784$; but $r+z$, the ſum of the 2 firſt terms of the ſeries is $=1.6876$, and therefore greater than the truth only by .0092.

Laſtly, if $r$ be taken equal to
3 , 4 , 5 , 6 , 7 , 8 , 9 & 10 ſuc.
then will | 2.6743 | 3.6720 | 4.6709 | 5.6702 | 6.6698 | 7.6695 | 8.6693 | 9.6692 | be
the value $s$ of $pn$ reſpectively; from whence it appears that ($n$) the required value, in all caſes where $p$ is very ſmall, will be $=\frac{r-.3}{p}$ nearly, or $pn=r-\frac{3}{10}$. Now from this, and the foregoing concluſion, where $p$ was ſuppoſed $=1$, and $pn$ found $=2r-1$, the following theorem is deduced; by help of which, the value of $n$ in any intermediate caſe may be obtained, it being always $=\frac{b}{a}\times\overline{r-.3}+r-\frac{7}{10}$, very nearly. Q. E. I.

## EXAMPLE I.

In how many throws with 3 common dice may one undertake to throw the 3 aces.

The number of chances for failing at any trial being 215, and for happening only one, $\frac{b}{a}$ will therefore be $=215$ and $\frac{b}{a}\times\overline{r-.3}+r-\frac{7}{10}=150$ the number that was to be found.

## EXAMPLE II.

In a lottery confiſting of a great number of tickets, where the blanks are to the prizes as 50 to 1; to find how many a perſon ought to take to expect 5 prizes?

Here $50\times\overline{5-.3}+5-.7=231$, is the anſwer.

## EXAMPLE III.

Suppoſe a lottery like the foregoing, where the blanks are to the prizes, as 3 to 1; to find how many muſt be taken to expect 8 prizes?

Here $r$ being 8, and $\frac{b}{a}=3$, $\frac{b}{a}\times\overline{r-.3}+r-.7$ becomes 30.4; therefore 30 or 31 is the anſwer.

## EXAMPLE IV.

In how many throws, with a ſingle die, may a perſon undertake to throw either the ace or duce?

Here $r$ being $=1$, $\frac{b}{a}=2$, $\frac{b}{a}\times\overline{r-.3}+r-.7$ will be 1.7, differing from the true value 1.709 *(found from the theorem in Prob.* VII.) by .009 only.

## PROBLEM XVI.

*Suppoſing a given Number (n) of Letters a, b, c, d, e, f, &c. or Things repreſented by them, to be placed in* Order, *and that a Perſon draws them one by one, as it happens, and lays them down in the Order they are taken; to find the Probability that any (p) aſſigned Letters ſhall have the very ſame Places in the ſecond as in the firſt Order, and (m) other aſſigned ones, at the ſame time, all different Places from what they have in that Order.*

SOLUTION.

## SOLUTION.

The probability that *b* happens not in the ſecond place of the ſecond order, and that *a* happens in the firſt place, and *b* out of the ſecond place of this order, are $1 - \frac{1}{n}$, and $\frac{1}{n} - \frac{1}{n \times \overline{n-1}}$ $\left(= \frac{1}{n} \times \frac{n-2}{n-1}\right)$ reſpectively (*by Prob.* I.) And therefore if the laſt of theſe be ſubtracted from the former, or, the probability that *b* ſhall not come into the ſecond place, without any reſtriction of having *a* either in or out of the firſt place, the remainder $1 - \frac{2}{n} + \frac{1}{n \times \overline{n-1}}$, it is manifeſt, will be the probability that neither *b* ſhall come in the ſecond place, nor *a* in the firſt; ſince *a* muſt neceſſarily be either in the firſt place or out of it; and the ſame is, alſo, the probability that any other 2 aſſigned letters ſhall happen in different places from what they poſſeſs in the firſt order. Wherefore if *a* ſhould be the firſt taken, of which the probability is $\frac{1}{n}$, then becauſe there would remain only $n - 1$ letters, the probability of *b* and *c* both happening out of their places (by ſubſtituting $n - 1$ for $n$, &c. in the ſaid expreſſion) would, it is manifeſt, be expreſſed by $1 - \frac{2}{n-1} + \frac{1}{\overline{n-1} \times \overline{n-2}}$; therefore this drawn into $\frac{1}{n}$ is $\frac{1}{n} - \frac{2}{n \times \overline{n-1}} + \frac{1}{n \times \overline{n-1} \times \overline{n-2}}$ equal to the probability that *a* ſhall fall in, and *b* and *c* out of their places; which being ſubtracted, in like manner, from $1 - \frac{2}{n} + \frac{1}{n \times \overline{n-1}}$, the probability of *b* and *a* both falling out of their places, without farther reſtriction, leaves $1 - \frac{3}{n} + \frac{3}{n\cdot\overline{n-1}} - \frac{1}{n\times\overline{n-1}\times\overline{n-2}}$ for the probability that *b*, *c* and *a*, or any other 3 aſſigned letters ſhall all happen out of their places. And therefore if *a* ſhould firſt happen to be taken, as there would then be only $n-1$ letters left, the probability that *b*, *c* and *d* would all happen out of their places is $1 - \frac{3}{\overline{n-1}} + \frac{3}{\overline{n-1}\times\overline{n-2}} - \frac{1}{\overline{n-1}\times\overline{n-2}\times\overline{n-3}}$; this, therefore multiplied by $\frac{1}{n}$, the probability of taking *a* firſt, will be the probability that *a* ſhall happen in its place, and *b*, *c* and *d* all out of their places; whence by ſubſtracting, as above, the probability of *a*, *b*, *c* and *d*, or any other 4 aſſigned letters, falling all out of their places appears to be $1 - \frac{4}{n} + \frac{6}{n\times\overline{n-1}} + \frac{4}{n\times\overline{n-1}\times\overline{n-2}} + \frac{1}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}$: and from hence the manner of the proceſs, and law of continuation are manifeſt; the numerators being the unciæ of the power of a binomical whoſe exponent is equal to the number of letters excluded their places; and the denominators $1$, $n$, $n\times\overline{n-1}$, $n\times\overline{n-1}\,\overline{n-2}$, &c. And therefore the probability that all the *m* aſſigned ones ſhall happen out of their places is $1 - \frac{m}{n} + \frac{m\times\frac{m-1}{2}}{n\times\overline{n-1}} - \frac{m\times\frac{m-1}{2}\times\frac{m-2}{3}}{n\times\overline{n-1}\times\overline{n-2}} + \frac{m\times\frac{m-1}{2}+\frac{m-2}{3}\times\frac{m-3}{4}}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}$ $(m+1)$: but the probability of any $p$ aſſigned letters, as $a$, $b$, $c$, &c. falling in their places is $\frac{1}{n\times\overline{n-1}\times\overline{n-2}\,(p)}$ (by *Prob.* II.); and if theſe letters ſhould be firſt ſo taken, the probability then of $m$ other aſſigned ones falling all out of their places, will be $1 - \frac{m}{n-p} + \frac{m\times\frac{m-1}{2}}{\overline{n-p}\times\overline{n-p-1}} - \frac{m\times\frac{m-1}{2}\times\frac{m-2}{3}}{\overline{n-p}\times\overline{n-p-1}\times\overline{n-p-2}}\,(m+1)$ *per* above, ($n$ the number of letters here becoming $n-p$): therefore $\frac{1}{n\times\overline{n-1}\times\overline{n-2}\,(p)}$ into $1 - \frac{m}{n-p} + \frac{m\times\frac{m-1}{2}}{\overline{n-p}\times\overline{n-p-1}} - \frac{m\times\frac{m-1}{2}\times\frac{m-2}{3}}{\overline{n-p}\times\overline{n-p-1}\times\overline{m-p-2}}$ $(m+1)$ muſt conſequently be the value. Q. E. I.

## COROLLARY I.

If $n-p$ be $=m$, or, the $p$ aſſigned letters be propoſed to be taken in, and all the reſt out of their places, the probability becomes $\frac{1}{n\times\overline{n-1}\times\overline{n-2}\,(p)}$ in $1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4}\,(m+1)$; which, when $m$ is a large number, is equal to $\frac{0.367878}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\,(p)}$ very nearly.

## COROLLARY II.

If a given number $p$ be to be taken in their places, and the reſt out of their places, without farther reſtriction, then the probability of taking any $p$ aſſigned ones in their places, and the reſt otherwiſe, being $\frac{1}{n\times\overline{n-1}\times\overline{n-2}}$ $(p)$ in $1 - 1 +$

$\frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4}\,(m+1)$; it is manifeſt, that as many different ways as $p$ things can be aſſigned, or taken in $n$ things, which is $n \times \frac{n-1}{2} \times \frac{n-2}{3}\,(p)$ ſo many times ought the ſaid quantity be repeated to give the probability in this caſe; which therefore will be $\frac{1}{1.2.3.4\,(p)}$ in. $1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4}\,(m+1)$; or, when $m$ is large $= \frac{0.367878}{1.2.3.4.5\,(p)}$ nearly.

## COROLLARY III.

Hence if $b$ be put $=0.367878$, and $p$ be taken $=0, 1, 2, 3$, &c. ſucceſſively, the ſaid probability, will, it is evident, become $b$, $b$, $\frac{b}{2}$, $\frac{b}{2.3}$, $\frac{b}{2.3.4}$, &c. reſpectively equal to the probability of taking preciſely 0, 1, 2, 3, &c. letters according to their places; wherefore if $b + b + \frac{b}{2}\,(p)$ be ſubtracted from unity the remainder $1 - b \times \overline{1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4}\,(p)}$ will be the probability, very nearly, that $p$, or a greater number of letters, fall according to their places in the firſt order.

## EXAMPLE I.

Let it be required to find how many changes can be rung on 5 bells, without any one ſtriking in its own proper place.

Here becauſe $p$ is $=0$, $n=5$, and $m=5$, the probability $\frac{1}{n \times \overline{n-1} \times \overline{n-2}\,(p)}$ in $1 - 1 + \frac{1}{2} - \frac{1}{2.3}\,(m+1)$, (*as in Cor.* I.) will become $1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} - \frac{1}{2.3.4.5} = \frac{44}{120}$; and therefore as all the changes on 5 bells are 120, the required number will be 44.

## EXAMPLE II.

A perſon holding 2 packs of cards, draws one out of one pack, and another out of the other, and puts them together, and does the like by 2 others, &c. repeating the operation till all the cards are exhauſted; what is the odds that the cards in one or more of thoſe couples ſhall be found the ſame?

In this caſe, the number (52) of things being large, 0.367878 will, either by Cor. I. or III. be the probability that the cards in no one couple ſhall be the ſame: therefore as 0.632122 is to 0.367878, ſo is the required odds that the cards in one or more of the couples ſhall be the ſame.

## PROBLEM XVII.

*Suppoſing a given Number ($r$) of each of ſeveral Sorts of Things, as aaa, bbb, ccc, ddd, eee, fff, &c. to be put together in Order, and afterwards drawn one by one at a Venture, and laid down in the Order they are taken; to find the Probability that any ($p$) aſſigned Sorts ſhall happen to have the ſame Places in the ſecond as in the firſt Order, and ($m$) other aſsigned Sorts, at the ſame Time, different Places from what they have in that Order.*

SOLUTION.

## SOLUTION.

This problem is ſolved by the ſame method of reaſoning as the foregoing; for if $n$ be put = the total number of things of all ſorts, and $1 \times 2 \times 3 \times 4 \times 5\ (r) = s$, it is evident, from Prob. III. that $\frac{s}{n \times \overline{n-1} \times \overline{n-2}\,(r)}$ will be the probability that all the *a's* are taken firſt; and therefore that of the contrary, or the probability that all the things of any aſſigned ſort ſhall not happen as they are in the firſt order, muſt be $1 - \frac{s}{n \times \overline{n-1}\,(r)}$. Therefore if the *a's* ſhould be all taken firſt, as there would be then only $n-r$ things left, the probability that the *b's* would not, all, come out next, or fall in their places, muſt, it is evident, be $1 - \frac{s}{\overline{n-r} \times \overline{n-r-1}\,(r)}$; wherefore this multiplied by $\frac{s}{n \times \overline{n-1}\,(r)}$, as expreſſed above, gives $\frac{s}{n \times \overline{n-1}\,(r)} - \frac{s^2}{n \times \overline{n-1}\,(2r)}$ for the probability that the $r$ firſt taken ſhall be all *a's*, and the $r$ next not all *b's*; and this ſubtracted from $1 - \frac{s}{n \times \overline{n-1}\,(r)}$, the probability that the *b's* ſhall not all be ſo taken, without any reſtriction whether the *a's* be, or be not, firſt taken, leaves $1 - \frac{2s}{n \times \overline{n-1}\,(r)} + \frac{s^2}{n \times \overline{n-1}\,(2r)}$ = the probability that neither the *a's* nor *b's* ſhall come out according to their places; whence, by repeating the operation in the ſame manner as in the laſt problem, the required probability will appear to be $\frac{s^{p}}{n\times\overline{n-1}\times\overline{n-2}\ (rp)}$ in. $1-\frac{ms}{\overline{n-rp}\times\overline{n-rp-1}\ (rp+r)}+\frac{m\times\frac{m-1}{2}ss}{\overline{n-rp}\times\overline{n-rp-1}\ (rp+2r)}$, &c. Where, if $r$ be taken $=1$, $s$ will be $=1$, and the ſolution the ſame as the foregoing.

## PROBLEM XVIII.

*Three Perſons,* A, B, *and* C, *throw in their Turns a Solid having* ($f$) *regular Faces, and he who firſt happens to bring up an aſſigned Face is thereon to be entitled to a certain Benefit; required the ſeveral Probabilities of obtaining it.*

## SOLUTION.

Let the value of the benefit or thing expected be denoted by unity: then ſince *(by Cor. to Prob.* I.) the probability that the aſſigned face ſhall fail $n-1$ throws ſucceſſively, and come up the next after, is $\frac{\overline{f-1}|^{n-1}}{f^{n-1}}\times\frac{1}{f}$, this quantity, it is manifeſt, will alſo expreſs the expectation on the throw whoſe number, from the beginning, is denoted by $n$: therefore in order to find the total expectation of *A*, as the 1ſt, 4th, 7th, &c. throws pertain to him, let $n$ be expounded by 1, 4, 7, 11, &c. ſucceſſively; then the above expreſſion will become $\frac{1}{f}$, $\frac{\overline{f-1}|^{3}}{f^{4}}$, $\frac{\overline{f-1}|^{6}}{f^{7}}$, &c. reſpectively, and the ſum of theſe will be the firſt value ſought. In like manner, if $n$ be expounded by 2, 5, 8, &c. and 3, 6, 9, &c. the total expectations, or probabilities of *B* and *C* will come out $\frac{f-1}{f^2}+\frac{\overline{f-1}|^4}{f^5}+\frac{\overline{f-1}|^7}{f^8}$, &c. and $\frac{\overline{f-1}|^2}{f^3}+\frac{\overline{f-1}|^5}{f^6}+\frac{\overline{f-1}|^8}{f^9}$, &c. reſpectively: but as theſe 3 values are infinite ſeries whoſe terms are in geometrical progreſſion, they may be very eaſily ſummed, being equal to $\frac{ff}{f^3-\overline{f-1}|^3}$, $\frac{f\times\overline{f-1}}{f^3-\overline{f-1}|^3}$ and $\frac{\overline{f-1}|^2}{f^3-\overline{f-1}|^3}$ reſpectively; which are the numbers that were required to be found. But the ſolution in any caſe may be more eaſily had by conſidering that the expectations of *A*, *B*, *C*, &c. on their firſt throws are to one another as their total expectations; for then by taking as many terms of the progreſſion $1$, $\frac{f-1}{f}$, $\frac{\overline{f-1}|^2}{f^2}$, $\frac{\overline{f-1}|^3}{f^3}$, &c. as there are players concerned, and dividing each of them by the whole ſum, the ſeveral quotients will be reſpectively equal to the required probabilities.

## PROBLEM XIX.

*Two Gameſters,* A *and* B, *whoſe Chances for winning any aſſigned Game are in the given Ratio of (a) to (b), enter into Play on this Condition; that* A *at the Beginning of every Game ſhall ſet the Sum (e) to the Sum (f), and that the Play ſhall laſt as long as he continues to win without Intermiſſion; it is required to find the Gain or Advantage of* A.

### SOLUTION.

## SOLUTION.

Since the expectation of *A* on any game, when it comes to be played, is $\frac{a}{a+b}\times\overline{e+f}$, if ($e$) his ſtake be deducted therefrom, the remainder $\frac{af-be}{a+b}$ will conſequently be his gain: therefore the advantage or gain on any game, whoſe number from the beginning is denoted by $n$, to be computed before the play begins, muſt be compounded of the ſaid gain $\frac{af-be}{a+b}$ and the probability that the play will not be ended before that game comes to be decided; that is, of $\frac{af-be}{a+b}$ and $\frac{a^{n-1}}{\overline{a+b}|^{n-1}}$ the probability of his winning all the preceding $n-1$ games: wherefore if $n$ be taken equal to 1, 2, 3, 4, &c. ſucceſſively, we ſhall have $\frac{af-be}{a+b}$ into 1, $\frac{a}{a+b}$, $\frac{a^2}{\overline{a+b}|^2}$, $\frac{a^3}{\overline{a+b}|^3}$, &c. for the gain on the 1ſt, 2d, 3d and 4th games, &c. reſpectively; that progreſſion, infinitely continued, or $\frac{af-be}{a+b}$ into $1+\frac{a}{a+b}+\frac{a^2}{\overline{a+b}|^2}+\frac{a^3}{\overline{a+b}|^3}$, &c. $=\frac{af}{b}-e$, muſt be the true value. Q. E. I.

## PROBLEM XX.

A *and* B, *whoſe Chances for winning any ſingle Game are in Proportion as* ($a$) *to* ($b$), *the former having* ($p$) *and the latter* ($q$) *Stakes, are determined to play together 'till one of them has loſt all:*

*all: to find their reſpective Probabilities of winning, with the Gain or Advantage of* A, *&c.*

## SOLUTION.

Let the expectation of $A$, when he has any given number $y$ of ſtakes in poſſeſſion, be expreſſed by $Q$; and when he has one ſtake more, or $y+1$, ſuppoſe his expectation to be encreaſed by $R$, or to become $Q+R$; and when he has yet one more, or $y+2$, let this laſt expectation be encreaſed by $S$, &c. &c. Now when he has $y+1$ ſtakes, he muſt after one game have either $y$ or $y+2$ ſtakes; if he wins that game he muſt have $y+2$, and his expectation will then be $Q+R+S$; this therefore multiplied by $\frac{a}{a+b}$, the probability of winning, gives $\frac{Qa+Ra+Sa}{a+b}$ for his expectation, in caſe of winning, while the event remains yet undetermined. If he loſes the game, he will have only $y$ ſtakes, and his expectation will then be $Q$; this multiplied by $\frac{b}{a+b}$, the probability of loſing, gives $\frac{bQ}{a+b}$ for the expectation in caſe of loſing: but it is manifeſt that $\frac{Qa+Ra+Sa+Qb}{a+b}$, the ſum of thoſe two, muſt be equal to $Q+R$, his total expectation in this circumſtance: whence by reduction we have $Sa=Rb$, or, $R$ to $S$, as $a$ to $b$; and hence it appears that the values of the aforeſaid increments, $R$, $S$, $T$, $V$, &c. are ſuch, that any one of them is to that which immediately ſucceeds it, in the given ratio of $a$ to $b$; and therefore muſt conſtitute a ſeries of terms in geometrical progreſſion, of which the common ratio is $\frac{b}{a}$. Wherefore, let $y$ be taken $=0$; then will $R+S+T\ (p)$, or its equal $R+\frac{Rb}{a}+\frac{Rb^2}{a^2}+\frac{Rb^3}{a^3}\ (p)$, it is manifeſt, be the total expectation of $A$ in the propoſed circumſtance; and for the ſame reaſon $R+\frac{Rb}{a}+\frac{Rb^2}{a^2}+\frac{Rb^3}{a^3}\ (p+q)$ will be his expectation, when he has $p+q$ ſtakes in poſſeſſion; but this by the queſtion is $=p+q$; wherefore $R$ equal $\frac{p+q}{1+\frac{b}{q}+\frac{b^2}{a^2}+\frac{b^3}{a^3}\ (p+q)}$; which ſubſtituted inſtead thereof in the other expreſſion gives $p+q$ into

$$\frac{1+\frac{b}{a}+\frac{b^2}{a^2}+\frac{b^3}{a^3}\ (p)}{1+\frac{b}{a}+\frac{b^2}{a^2}+\frac{b^3}{a^3}\ (p+q)}=\overline{p+q}\times a^q\times\frac{a^p-b^p}{a^{p+q}-b^{p+q}}$$

for the expectation of $A$ in the propoſed circumſtance; which ſubtracted from $p+q$ leaves $\overline{p+q}\times b^p\times\frac{a^q-b^q}{a^{p+q}-b^{p+q}}$ for that of $B$; therefore the gain of $A$ is $-p+\overline{p+q}\times a^q\times\frac{a^p-b^p}{a^{p+q}-b^{p+q}}$, and the ratio of the probabilities of winning as $a^q\times\overline{a^p-b^p}$ to $b^p\times\overline{a^q-b^q}$, reſpectively. Q. E. I.

## COROLLARY I.

When $a$ is $=b$, then $R+\frac{Rb}{a}+\frac{Rb^2}{a^2}+\frac{Rb^3}{a^3}\ (p)$ will be $=pR$; and the odds directly as the ſtakes to loſe.

## COROLLARY II.

If $p=q$, then will $a^q \times \overline{a^p - b^p}$, to $b^p \times \overline{a^q - b^q}$, become as $a^p$ to $b^p$, for the odds in this caſe.

## COROLLARY III.

When $b$ is conſiderably ſmaller than $a$, and $p$ and $q$ large numbers, $b^p$ and $b^q$ will be inconſiderable in reſpect of $a^p$ and $a^q$; and therefore in that caſe the odds will be as $a^p$ to $b^p$, or as 1 to $\left.\frac{b}{a}\right|^p$ nearly.

## EXAMPLE.

Suppoſe $B$ to be throwing with 2 dice, and every time that 2 aces come up, $A$ to give him 1 guinea, and every time that an ace and duce, $B$ to give $A$ one, and that they agree to continue to play on in this manner till one of them is a winner of 100 guineas; to find the expectation of each, and the advantage or gain of $A$ by this agreement.

Becauſe there are 2 chances for an ace and duce, and only 1 for 2 aces, $a$ will here $= 2$, and $b = 1$, and therefore the odds *(by Cor.* II.) as $2^{100}$, to $1^{100}$, or as 1267000000000000000000000000000, to 1 nearly; whence the expectation of $A$ will be $\frac{200 \times 2^{100}}{2^{100}+1}$, and that of $B$, $\frac{200}{2^{100}+1}$ which is not $\frac{1}{10000000000000000, \&c.}$ part of a farthing; therefore the gain of $A$ is 100 guineas leſſened by that ſmall quantity.

PROBLEM.

## PROBLEM XXI.

*Two Gameſters,* A *and* B, *are at play together, and the latter having loſt* ($p$) *Stakes, is determined not to give out, till he has won them again; to find the Probability that he never effects his Deſire, ſuppoſing the Play to continue without Limitation; his Number of Chances* ($b$) *for winning any aſſigned Game being leſs than* ($a$) *that for the contrary.*

## SOLUTION.

It appears by the laſt problem that the odds of *A's* winning $q$ ſtakes before he loſes $p$ ſtakes will be as $a^q\times\overline{a^p-b^p}$ to $b^p\times\overline{a^q-b^q}$: therefore, becauſe *B*, if he ever wins his $p$ ſtakes again, muſt do it before he is a loſer of an infinite number of other ſtakes, let $q$ in the above proportion be ſuppoſed infinite, then it will ſhew the odds in this caſe; but then as $b^q$ will bear no compariſon with $a^q$, it will be as $a^q\times\overline{a^p-b^p}$, to $b^p\times a^q$, that is, as $a^p-b^p$, to $b^p$; therefore the required probability is $1-\frac{b}{a}\Big|^{p}$.

## PROBLEM XXII.

*To find the Chances that there are for throwing preciſely any Number of Points* ($p$), *with any Number of Dice* ($n$), *each Die having a given Number* ($f$) *of Faces.*

## SOLUTION.

Firſt, let there be a ſet of dice, having each $p$, or a greater number of faces; then the chances for $p$, and all its inferior numbers, on $n$ ſuch dice, will be equal to the chances for throwing $p+1$ points preciſely with $n+1$ of the ſame dice; ſince it is evident that with all the chances for $p$ on the $n$ firſt dice, the ace of the new added die may be combined, and with all the chances for $p-1$ the duce of the ſame, &c. And therefore, if in the annexed ſcheme, 1, 2, 3, or 4, &c. be put to denote the number of dice, and $p-2$, $p-1$, $p$, or $p+1$, &c. the number of points, and the quantities in the interſection of the columns be the chances for throwing the ſaid points with thoſe dice; that is, if for throwing $p-1$ points with 3 dice the chances be $C'$, for $p$ points with 4 dice, $D$, &c. &c. it is manifeſt, that $B_{\prime}$ will $=A+A'+A''+A'''$ &c. $D_{\prime}=C+C'+C''$, &c. $D=C'+C''$, &c. and therefore $D_{\prime}-D=C$, and conſequently, for the ſame reaſon, $E_{\prime}-E=D$, $F_{\prime}-F=E$, &c. &c. Wherefore it appears that the values of $B$, $C$, &c. are ſuch, that increaſing $p$ by unity they will be augmented by 1, $A$, $B$, &c. Whence by the method of increments thoſe values are eaſily had equal to $p-1$, $\frac{p-1}{1}\times\frac{p-2}{2}$, $\frac{p-1}{1}\times\frac{p-2}{2}\times\frac{p-3}{3}$, &c. reſpectively. Therefore it is manifeſt, that the number of chances for throwing $p$ points preciſely with $n$ ſuch

|  | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| &c. | &c. | &c. | &c. | &c. | &c. | &c. | &c. |
| $p-2$ | $A''$ | $B''$ | $C''$ | $D''$ | $E''$ | $F''$ | $G''$ |
| $p-1$ | $A'$ | $B'$ | $C'$ | $D'$ | $E'$ | $F'$ | $G'$ |
| $p$ | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ |
| $p+1$ | $A_{\prime}$ | $B_{\prime}$ | $C_{\prime}$ | $D_{\prime}$ | $E_{\prime}$ | $F_{\prime\prime}$ | $G_{\prime}$ |

ſuch dice will be $\frac{p-1}{1} \times \frac{p-2}{2} \times \frac{p-3}{3}$ $(n-1)$. But now in order to find the required value from hence, let the chances expreſſed by this ſeries be called *S*, and any one aſſigned die *A*, another *B*, a third *C*, &c. and ſuppoſe the points on each face where the number is greater than *f* to be red, and the reſt black ones; and, for the ſake of perſpicuity, let the red or black faces of any die be called the red or black part of that die: this being premiſed, it is evident that the chances for *p* points preciſely with *n* of theſe dice, ſo as to have the red part of the die *A*, always upwards, will be exactly equal to the chances on the ſame dice for $p-f$, without any reſtriction. For let the points on each red face of that die be conceived to be diminiſhed by *f*, then as one or other of the faces ſo diminiſhed is, in this caſe, always upwards, the number of chances for *p* before ſuch diminution will conſequently be equal to the chances for $p-f$ after it, that is, if $p-f$ be put $=q$, equal to $\frac{q-1}{1} \times \frac{q-2}{2} \times \frac{q-3}{3}$ $(n-1)$ by what has juſt now been determined. And from the ſame way of reaſoning it is plain that the number of chances for *p* points, ſo as to have the red parts both of *A* and *B* always upwards, is $\frac{r-1}{1} \times \frac{r-2}{2} \times \frac{r-3}{3}$ $(n-1)$: and for throwing the ſame points, ſo as to have the red parts of *A*, *B*, and *C* upwards, $\frac{s-1}{1} \times \frac{s-2}{2} \times \frac{s-3}{3}$ $(n-1)$ &c. &c. *r*, *s*, *t*, &c. being equal to $p-2f$, $p-3f$, $p-4f$, &c. reſpectively. Now let theſe quantities $\frac{q-1}{1}\times\frac{q-2}{2}\times\frac{q-3}{3}\,(n-1)$, $\frac{r-1}{1}\times\frac{r-2}{2}\times\frac{r-3}{3}\,(n-1)$, &c. be reſpectively denoted by *G*, *H*, *I*, *K*, &c. And let the chances in *S*, where the red part of *A* is up alone without other red ones, be called *g*; thoſe, where the red parts of *A* and *B* are up together without others of the ſame colour, *h*; thoſe, where the red parts of *A*, *B*, and *C* are all up together without other red ones, *i*, &c. &c. Then, it is evident, that *g* is alſo the number of chances for having the red part of *B* alone, or of *C* alone, &c. and therefore all the chances in *S* for having one red face preciſely will be $ng$: alſo as *h* is the chances wherein either *A C*, *A D*, *B C*, or *C D*, &c. are red, and the reſt black; and becauſe theſe, the combinations of *n* things taken 2 by 2, are in number $=n\times\frac{n-1}{2}$, the chances in *S*, with 2 red faces preciſely, in each, will therefore be $n\times\frac{n-1}{2}h$; and furthermore, ſince the number of combinations of *n* things taken 3 by 3, 4 by 4, &c. is $n\times\frac{n-1}{2}\times\frac{n-2}{3}$, and $n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$, &c. it follows that the ſum of all the chances in *S*, with one or more red faces, in each, will be $ng+n\times\frac{n-1}{2}h+n\times\frac{n-1}{2}\times\frac{n-1}{3}i$, &c. which ſubtracted from *S*, the whole number of chances, leaves $S-ng-n\times\frac{n-1}{2}h-n\times\frac{n-1}{2}\times\frac{n-2}{3}i-n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}k$, &c. for the number of chances with black points alone which is manifeſtly equal to the number ſought. But now to get rid of $g$, $h$, &c. let $Q$ be the chances in $S$ that are made with the red parts of any number $a$ of aſſigned dice, as $A$, $B$, $C$, &c. without other parts of the ſame colour; or, which is the ſame, let $Q$ denote any one of the ſaid quantities $g$, $h$, &c. and let $p$ denote the chances in $S$, having in each of them, alſo, preciſely $a$ red parts, but only a given number $c$ of them aſſigned ones, the other $a-c$ being varied as often as poſſible; then as the number of dice or parts to be thus varied, or combined, is $n-c$, the number of ſuch combinations will be $\frac{n-c}{1} \times \frac{n-c-1}{2} \times \frac{n-c-2}{3}$ $(a-c)$ *by Prob.* III. And therefore the number of chances in the latter of theſe 2 caſes, juſt $\frac{n-c}{1} \times \frac{n-c-1}{2} \times \frac{n-c-2}{3}$ $(a-c)$ times as great as that in the former, or, $P = Q$ into $\frac{n-c}{1} \times \frac{n-c-1}{2} \times \frac{n-c-2}{3}$ $(a-c)$. Now taking $c=1$, and $a$ equal to 1, 2, 3, 4, &c. ſucceſſively, $Q$ becomes $g$, $h$, $i$, $k$, &c. and $P$ equal to $g$, $\frac{n-1}{1} h$, $\frac{n-1}{1} \times \frac{n-2}{2} i$, $\frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} k$, &c. reſpectively; therefore as theſe quantities reſpectively ſhew the chances in $S$, having the red part of $A$ alone, or with 1 other, or 2 others, &c. preciſely, their ſum muſt conſequently be equal to $G$, all the chances in $S$ with the red part of $A$, and therefore $g = G - \frac{n-1}{1} h - \frac{n-1}{1} \times \frac{n-2}{2} i$, &c.

$i$, &c. In like manner by taking $c=2$, and $a=2, 3, 4$, &c. ſucceſſively, &c. we have $h=H-\frac{n-2}{1}i-\frac{n-2}{1}\times\frac{n-3}{2}k$, &c. $i=I-\frac{n-3}{1}k-\frac{n-3}{1}\times\frac{n-4}{2}l$, &c. Whence by ſubſtituting theſe values one by one in $S-ng-n\times\frac{n-1}{2}h-n\times\frac{n-1}{2}\times\frac{n-2}{3}i$, &c. as above found, we have $S-nG+n\times\frac{n-1}{2}H-n\times\frac{n-1}{2}\times\frac{n-2}{3}I+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}K$, &c. where by reſuming the known values of $S$, $G$, $H$, &c. there will be

$$+\frac{p-1}{1}\times\frac{p-2}{2}\times\frac{p-3}{3}(n-1)$$
$$-\frac{q-1}{1}\times\frac{q-2}{2}\times\frac{q-3}{3}(n-1)\text{ in. }n$$
$$+\frac{r-1}{1}\times\frac{r-2}{2}\times\frac{r-3}{3}(n-1)\text{ in. }n\times\frac{n-1}{2}$$
$$-\frac{s-1}{1}\times\frac{s-2}{2}\times\frac{s-3}{3}(n-1)\text{ in. }n\times\frac{n-1}{2}\times\frac{n-2}{3}$$

&c. where $q$ is $=p-f$, $r=p-2f$, $s=p-3f$, &c. and the number ſeries to be continued till ſome one factor becomes nothing or negative. Q. E. I.

## COROLLARY.

Hence the chances for not coming up a greater number of points than $p$ at any aſſigned throw is very readily determined, being equal to

$$p\times\frac{p-1}{2}\times\frac{p-2}{2}(n)$$
$$-q\times\frac{q-1}{2}\times\frac{q-2}{3}(n)\text{ in }n$$

$$r \times \frac{r-1}{2} \times \frac{r-2}{3}\ (n) \text{ in } n \times \frac{n-1}{2}$$

&c. &c.

## EXAMPLE I.

Let it be required to find the odds that a perſon at the firſt throw with 2 common dice does not throw juſt 8 points; then $f$ being $=6$, $p=8$, and $n=2$, the general expreſſion becomes $=7-2=5=$ the number of chances for 8 points, which therefore taken from $6^2$, the whole number of all the chances, leaves 31 chances for the contrary; whence the required odds is as 31 to 5.

Note, *When the number of points propoſed to be thrown, according to the Problem, is nearer to the greateſt number that can be had, than the leaſt, it will be convenient to uſe inſtead thereof that number, as far diſtant from the leſſer extreme, as it is from the greater, the chances for both being manifeſtly the ſame.*

## EXAMPLE II.

How many chances are there to have 50 points, preciſely, on 10 dice? Here 20 being as much greater than one extreme (10) as 50 is exceeded by the other (60), we ſhall have $p=20$, $n=10$, $f=6$, $q=14$, $r=8$, $s=2$; and therefore $\frac{19}{1} \times \frac{18}{2} \times \frac{17}{3} \times \frac{16}{4} \times \frac{15}{5} \times \frac{14}{6} \times \frac{13}{7} \times \frac{12}{8} \times \frac{11}{9} - \frac{13}{1} \times \frac{12}{2} \times \frac{11}{3} \times \frac{10}{4} \times \frac{9}{5} \times \frac{8}{6} \times \frac{7}{7} \times \frac{6}{8} \times \frac{5}{9} = 85228$, the number that was to be found.

## EXAMPLE III.

What are the odds that at the firſt throw with 10 dice there ſhall come up more than 15 points? Here taking $n=10$, $f=6$, $p=15$, and ſubſtituting theſe values in the ſecond general expreſſion, we ſhall have 3003 for the number of chances by which 15 points and all its inferior numbers may happen; this taken from $60466176=\overline{6}|^{10}$ the whole number of chances leaves 60463173; therefore as 60463173 to 3003, ſo are the odds.

A Table exhibiting the Chances by which any Number of Points may be preciſely had with 10 common Dice.

| Points. | Chan. | Points. | Chan. | Points. | Chances. | Points. | Chances. |
|---|---|---|---|---|---|---|---|
| 10 · 60 | 1 | 17 · 53 | 11340 | 24 · 46 | 576565 | 31 · 39 | 3393610 |
| 11 · 59 | 10 | 18 · 52 | 23760 | 25 · 45 | 831204 | 32 · 38 | 3801535 |
| 12 · 58 | 55 | 19 · 51 | 46420 | 26 · 44 | 1151370 | 33 · 37 | 4121260 |
| 13 · 57 | 220 | 20 · 50 | 85228 | 27 · 43 | 1535040 | 34 · 36 | 4325310 |
| 14 · 56 | 715 | 21 · 49 | 147940 | 28 · 42 | 1972630 | 35 · 35 | 4395456 |
| 15 · 55 | 2002 | 22 · 48 | 243925 | 29 · 41 | 2446300 | Total of all the chances being $\overline{6}\vert^{10}$ $=60466176$. | |
| 16 · 54 | 4995 | 23 · 47 | 383470 | 30 · 40 | 2930455 | | |

## PROBLEM XXIII.

*There is a Solid having (m) ſimilar and equal Faces, whereof (p) are marked A; (q) of them, B; r, C, &c. What is the Probability that in throwing up a given Number (n) of ſuch Solids there ſhall ariſe a given Number (b) of aſſigned Sorts of Faces, as A, B, C, &c.*

### SOLUTION.

## SOLUTION.

The probability that no *A* ſhall come up being $\frac{\overline{m-p}|^{n}}{m^{n}}$, (*by Prob.* I.) that of the contrary muſt be $\frac{m^{n}-\overline{m-p}|^{n}}{m^{n}}$; and therefore as the whole number of chances is $m^{n}$, that for having one, or more *A* upwards will conſequently be $m^{n}-\overline{m-p}|^{n}$; that is, there are $m^{n}-\overline{m-p}|^{n}$ different ways by which the faces of the ſolids may be varied to have one *A* or more up at each variation. Therefore if all the faces marked *B* be now reſtrained from coming up, then there being only $m-q$ faces that can ariſe, the laſt expreſſion will (by ſubſtituting $m-q$ inſtead $m$), it is evident, become $\overline{m-q}|^{n}-\overline{m-p-q}|^{n}$, equal to all the variations that can poſſibly be made to have *A* up, when *B* is reſtrained from appearing; which therefore being ſubtracted from $m^{n}-\overline{m-p}|^{n}$, the chances for having *A* up, without any reſtriction upon *B*, leaves $m^{n}-\left|\begin{matrix}\overline{m-p}|^{n}\\ \overline{m-q}|^{n}\end{matrix}\right.+\overline{m-p-q}|^{n}$ for the number of chances to have both *A* and *B* upwards; ſince it is ſelf evident that in all the ſaid variations for *A*, either *B* muſt, or muſt not, be upwards. In like manner, if the *C's* be all reſtrained from coming upwards, $m$, the number of faces in each ſolid, may be conceived to be reduced to $m-r$, and then the laſt expreſſion will become $\overline{m-r}|^{n}-\left|\begin{matrix}\overline{m-p-r}|^{n}\\ \overline{m-q-r}|^{n}\end{matrix}\right.+\overline{m-p-q-r}|^{n}$ for the number of variations that can be made with *A* and *B* up, when *C* is reſtrained from appearing; wherefore this taken from $m^{n}-\left|\begin{matrix}\overline{m-p}|^{n}\\ \overline{m-q}|^{n}\end{matrix}\right.+\overline{m-p-q}|^{n}$, as found above, muſt

leave $$m^n - \left|\begin{array}{l}\overline{m-p}|^n \\ \overline{m-q}|^n \\ \overline{m-r}|^n\end{array}\right| + \left|\begin{array}{l}\overline{m-p-q}|^n \\ \overline{m-p-r}|^n \\ \overline{m-q-r}|^n\end{array}\right| - \overline{m-p-q-r}|^n,$$
for the number of different ways by which *A*, *B* and *C* may be all upwards. By the ſame method of reaſoning the chances for having *A*, *B*, *C* and *D* all upwards will be found

$$m^n - \left|\begin{array}{l}\overline{m-p}^n \\ \overline{m-q}^n \\ \overline{m-r}|^n \\ \overline{m-s}|^n\end{array}\right| + \left|\begin{array}{l}\overline{m-p-q}|^n \\ \overline{m-p-r}|^n \\ \overline{m-p-s}|^n \\ \overline{m-q-r}|^n \\ \overline{m-q-s}|^n \\ \overline{m-r-s}|^n\end{array}\right| - \left|\begin{array}{l}\overline{m-p-q-r}|^n \\ \overline{m-p-q-s}|^n \\ \overline{m-p-r-s}|^n \\ \overline{m-q-r-s}|^n,\end{array}\right| + \overline{m-p-q-r-s}|^n$$

whence the law of continuation is manifeſt. Wherefore dividing the quantity thus reſulting by $m^n$, the whole number of chances, the quotient, it is manifeſt, will be the probability required.

## COROLLARY I.

Wherefore, when *p*, *q*, *r*, &c. are equal to one another, the number of chances in the laſt caſe will. it is evident. become $m^n - 4 \times \overline{m-p}|^n + 6 \times \overline{m-2p}|^n - 4 \times \overline{m-3p}|^n + \overline{m-4p}|^n$; where the unciæ are thoſe of a binomial raiſed to the power whoſe index is equal to the number of ſorts *A*, *B*, *C*, *D*, whoſe chances for happening all upwards, are exhibited by that expreſſion; and the like appears in any other caſe. Therefore when *p*, *q*, *r*, &c. are as above ſpecified,

$$\frac{m^n - b \times \overline{m-p}|^n + b \times \frac{b-1}{2} \times \overline{m-2p}|^n - b \times \frac{b-1}{2} \times \frac{b-2}{3} \times \overline{m-3p}|^n}{m^n}$$

&c.

&c. muſt conſequently be the value propoſed to be found.

## COROLLARY II.

Becauſe when $m$ is a large, and $p$ a ſmall number, the quantities $m^n$, $\overline{m-p}|^n$, $\overline{m-2p}|^n$, &c. are nearly in a geometrical progreſſion, or equal to $m^n$, $\overline{m-p}|^n$, $\frac{\overline{m-p}|^{2n}}{m^n}$, &c. reſpectively, the above expreſſion, in this caſe, will be nearly equal to $\frac{m^n - b\times\overline{m-p}|^n + b\times\frac{b-1}{2}\times\frac{\overline{m-p}|^{2n}}{m^n}, \text{ \&c.}}{m^n} = 1 - b\times\overline{1-\frac{p}{m}}|^n + b\times\frac{b-1}{2}\times\overline{1-\frac{p}{m}}|^{2n} - b\times\frac{b-1}{2}\times\frac{b-2}{3}\times\overline{1-\frac{p}{m}}|^{3n}$, &c. or that power of the binomial $1-\overline{1-\frac{p}{m}}|^n$ whoſe index is $b$, that is $=\overline{1-\overline{1-\frac{p}{m}}|^n}|^h$; and therefore if the required probability be denoted by $P$, in this caſe $P$ will $=\overline{1-\overline{1-\frac{p}{m}}|^n}|^h$ very nearly: whence $P^{\frac{1}{h}} = 1 - \overline{1-\frac{p}{m}}|^n$, $\overline{1-\frac{p}{m}}|^n, = 1-P^{\frac{1}{h}}$, and $n = \frac{\text{Log. } \overline{1-P^{\frac{1}{h}}}}{\text{Log. } \overline{1-\frac{p}{m}}} = -\frac{p}{m}$ in hyp. log. $\overline{1-P^{\frac{1}{h}}}$ nearly; by which, if $P$ be given, $n$ may be obtained.

## EXAMPLE I.

One with 2 common dice undertakes to throw both the numbers 5 and 7 at 3 trials; what are the odds againſt him? Here if we ſuppoſe a die, or ſolid, having 36 faces, whereof 4 are marked *A*, and 6 *B*; it is manifeſt that the probability of throwing both *A* and *B* in 3 throws with that ſolid, is the ſame as that of throwing both the numbers 5 and 7 at 3 trials with 2 common dice: wherefore, according to the general theorem we have $m=36$, $p=4$, $q=6$, $n=3$, and $\frac{m^n - \overline{m-p}|^n - \overline{m-q}|^n + \overline{m-p-q}|^n}{m^n} = \frac{18^3 - 16^3 - 15^3 + 13^3}{\overline{18}|^3} = \frac{31}{324}$; whence the odds are as 293 to 31.

## EXAMPLE II.

Let it be required to find the probability that a perſon in throwing 6 common dice ſhall bring up the ace, duce and tray.

In this caſe, $n$ is $=6$, $b=3$, $m=6$, $p=1$, $q=1$, &c. Therefore *(by Cor.* I.) $\frac{6^6 - 3 \times 5^6 + 3 \times 4^6 - 3^6}{6^6} = \frac{35}{144}$ will be the value ſought.

## EXAMPLE III.

To find in how many throws with a ſingle die one may undertake to throw all the 6 faces.

By comparing this caſe with Cor. II. we have $n=1$, $m=6$, $b=6$, $P=\frac{1}{2}$, and $(n)$ the required number equal $\frac{\text{Log. } \overline{1-\frac{1}{2}}|^{\frac{1}{6}}}{\text{Log. } 1-\frac{1}{6}} = 12$.

## PROBLEM XXIV.

*To find the Probability that a propoſed Event ſhall happen a given Number of Times (p) without Intermiſſion in a given Number (n) of Tryals.*

Let *r* be the probability of happening of the propoſed event at any aſſigned trial, and *m* that of the contrary; and upon the happening of the ſaid event *p* times ſucceſſively, let a perſon, *B*, receive a certain ſum *S*: then will the probability of receiving that ſum at the end of an aſſigned number of trials be compounded of 3 others; as, firſt, the probability of happening of the propoſed event *p* times in ſo many trials: ſecond, that of its failing the time immediately preceding thoſe trials: and laſtly, that of his not having received it before that time. For if the firſt ſucceed not, the thing is manifeſtly impoſſible; if the ſecond prove contrary, he muſt either not recieve it at all, or at the end of ſome of the preceding trials; and, laſtly, when he has once received it, there can be no farther probability of obtaining it from any future trial.

Therefore, as the probability of the propoſed event's failing any one aſſigned trial, and then happening *p* times without intermiſſion, is $m=r^p$, (*by Prob.* I.) if $r^p$ be put $=a$, and $m\times r^p=x$, we ſhall have $x$ equal to the probability of his receiving the ſaid ſum at the end of $p+1$, $p+2$, $p+3$ ..... or $p+p$ trials, becauſe, it is manifeſt, the laſt of the aforenamed probabilities does not take place till after the $2p$ firſt trials: but the manner how each value is derived from the preceding ones, and the relation they bear to one another, will appear better by help of the following ſcheme; wherein the ſecond column towards the left hand ſhews the probability of receiving that ſum at the end of a given number of trials repreſented by the firſt; and the third, the probability that he receives it ſome time in thoſe trials. Each line, or expreſſion, of the ſecond column being formed by drawing $x$ into the exceſs of unity above the value of that line of the third column, whoſe diſtance, or place above the line ſo formed, is denoted by $p+1$; according to the reaſons above ſpecified, this laſt column being generated by a continual addition of the terms of the former.

| Tryals. | Probability. | Probability. |
|---|---|---|
| $p$ | $a$ | $a$ |
| $p+1$ | $x$ | $a+x$ |
| $p+2$ | $x$ | $a+2x$ |
| &c. | $x$ | &c. |
| $2p$ | $x$ | $a+px$ |
| $2p+1$ | $x\times\overline{1-a}$ | $a-ax+\overline{p+1}\times x$ |
| $2p+2$ | $x\times\overline{1-a-x}$ | $a-2ax+\overline{p+2}\times x-x^2$ |
| $2p+3$ | $x\times\overline{1-a-2x}$ | $a-3ax+\overline{p+3}\times x-3x^2$ |
| $2p+4$ | $x\times\overline{1-a-3x}$ | $a-4ax+\overline{p+4}\times x-6x^2$ |
| $3p$ | $x\times\overline{1-a}-\overline{p-1}\times x$ | $a-pax+2px-p\times\frac{p-1}{2}x^2$ |

Now from having proceeded thus far, the law of continuation is manifeſt; the value of the 3d column, or the required probability, in any caſe, (where $n$ is not leſs than $p$) being $a$ in $1-$

$$\overset{''}{n}x+\overset{'''}{n}\times\frac{\overset{'''}{n}-1}{2}x^2-\overset{''''}{n}\times\frac{\overset{''''}{n}-1}{2}\times\frac{\overset{''''}{n}-2}{3}x^3+\overset{v}{n}\times\frac{\overset{v}{n}-1}{2}\times\frac{\overset{v}{n}-2}{3}\times\frac{\overset{v}{n}-3}{4}$$

$\frac{\overset{v}{n}-3}{4}x^4$, &c. Plus $\overset{\prime}{n}x-\overset{\prime\prime}{n}\times\frac{\overset{\prime\prime}{n}-1}{2}x^2+\overset{\prime\prime\prime}{n}\times\frac{\overset{\prime\prime\prime}{n}-2}{3}x^3-\overset{\prime\prime\prime\prime}{n}\times\frac{\overset{\prime\prime\prime\prime}{n}-1}{2}\times\frac{\overset{\prime\prime\prime\prime}{n}-2}{3}\times\frac{\overset{\prime\prime\prime\prime}{n}-3}{4}x^4$, &c. where $\overset{\prime}{n}$ is put $=n-p$, $\overset{\prime\prime}{n}=n-2p$, $\overset{\prime\prime\prime}{n}=n-3p$, &c. Q. E. I.

## EXAMPLE I.

Let it be required to throw a propoſed chance 3 times without intermiſſion in 10 trials; when the odds for its happening any aſſigned trial are as 2 to 1: then taking $r=\frac{2}{2+1}$, $m=\frac{1}{2+1}$, $p=3$, $n=10$; we have $\overset{\prime}{n}=7$, $\overset{\prime\prime}{n}=4$, $a=\frac{8}{27}$, $x=\frac{8}{81}$, and therefore $\frac{8}{27}=\overline{1-\frac{32}{81}}+\frac{56}{81}-\frac{128}{27\times81}=\frac{592}{729}$ for the probability in this caſe.

## EXAMPLE II.

In 200 throws with a ſingle die, what are the odds that the ace does not come up 8 times ſucceſſively? Here $r$ being $=\frac{1}{6}$, $m=\frac{5}{6}$, $p=8$, $n=200$, our ſeries become $\frac{1}{6^8}\times1-\frac{5\times184}{6^9}$, &c. $+\frac{5\times192}{6^9}-\frac{92\times183\times25}{6^{18}}$, &c. $=.0000966442$ nearly; which ſubtracted from 1, leaves .9999033558; therefore the required odds will be as .999903 &c. to .0000966442, or as 10356 to 1 nearly.

Note. *Though to have the anſwer accurately true, both the ſeries ought to be continued till they terminate, or ſome of the factors become negative; yet*

*yet if a near approximation be only wanted, a few of the firſt terms may ſuffice as in the laſt example, where by taking only 2 terms of each ſeries, the true ſolution is exhibited to leſs than* $\frac{1}{10000}$ *part of the whole.*

## PROBLEM XXV.

A *and* B, *whoſe Proportion of Skill, or Chance for winning any aſſigned Game, is as (a) to (b), agree to play together, till the former is a Winner of (p) Stakes, or the latter of (q) Stakes; to find the Probability, that the Play between them ends in a given Number (n) of Games.*

### SOLUTION.

CASE I. *Let* $p=2$ *and* $q=2$; then the required value will be $1-\frac{\overline{2ab}|^{\frac{n}{2}}}{\overline{a+b}|^{n}}$ when $n$ is an even number, and $1-\frac{\overline{2ab}|^{\frac{n-1}{2}}}{\overline{a+b}|^{n-1}}$ when an odd one.

CASE II. *Let* $p=3$ *and* $q=3$; then $1-\frac{\overline{3ab}|^{\frac{n-1}{2}}}{\overline{a+b}|^{n-1}}$, or $1-\frac{\overline{3ab}|^{\frac{n-2}{2}}}{\overline{a+b}|^{n-2}}$, according as $n$ is odd or even, will be the value in this caſe.

CASE III. *Suppoſing* a *and* b, *or the Chances of the Gameſters to be equal.* From the binomial $1+1$ raiſed to the $n$ power, cut off as many terms as there are units in $\frac{n-p+\frac{3}{2}\pm\frac{1}{2}}{2}$, and of theſe take the $q$ laſt, reject the $p$ preceding, take the $q$ next, reject the $p$ next, &c. till all the terms are exhauſted; then the ſum of all the terms thus taken, divided by $2^{n-1}$, will be the probability of the play ending in favour of $A$, in the propoſed number of games, if $n-p$ be an odd number. And if the terms of thoſe cut off, whoſe diſtances from the laſt are denoted by 0, $q$, $q+p$, $2q+p$, $2q+2p$, $3q+2p$, &c. be taken with ſigns $-$ and $+$ alternately, and added to twice the above ſaid ſum, and the whole be divided by $2^n$, you will have the probability of the ſame in the other caſe when $n-p$ is not an odd number.

*Generally.* Put $\frac{n-p+\frac{1}{2}\mp\frac{1}{2}}{2}=m$, and let the $m$ firſt terms of $\overline{a+b}|^{n}$ expreſſed in a ſeries be denoted by $R$; the $m-q$ firſt, by $S$; the $m-q-p$ firſt, by $T$; the $m-2q-p$ firſt, by $U$, &c. Alſo let $l=\frac{n-p+\frac{1}{2}\pm\frac{1}{2}}{2}$, and the $l$ laſt terms of the ſaid ſeries be denoted by $r$; the $l-q$ laſt, by $s$; the $l-q-p$ laſt, by $t$: the $l-2q-p$ laſt, by $v$, &c. Then the probability of the play ending in favour of $A$, in the given number of games, will be $R-S\times\overline{\frac{b}{a}}|^{q}+T\times\overline{\frac{b}{a}}|^{q+p}-U\times\overline{\frac{b}{a}}|^{2q+p}+W\times\overline{\frac{b}{a}}|^{2q+2p}$, &c. $+r\times\overline{\frac{a}{b}}|^{p}-s\times\overline{\frac{a}{b}}|^{p+q}+t\times\overline{\frac{a}{b}}|^{2p+q}-v\times\overline{\frac{a}{b}}|^{2p+2q}+w\times\overline{\frac{a}{b}}|^{3p+2q}$, &c. when the whole is divided by $\overline{a+b}|^{n}$. And it is manifeſt, that what is ſaid in either of the 2 laſt caſes, will hold equally, in reſpect to the play ending in favour of $B$, if $p$ be changed for $q$, $q$ for $p$, $b$ for $a$, and $a$ for $b$.

## COROLLARY.

If $q$, the number of ſtakes which $A$ has to loſe, be greater than $n$ the number of games, then will $S=0$, $T=0$, &c. and the general expreſſion $=R+\frac{ra^p}{b^p}$; which ſhews the probability that he has of being a winner of $p$ ſtakes in the given number of games, when the duration of the play is not reſtrained by what he may happen to loſe: and this, when $n-p$ is an odd number, and $a=b$, will, it is manifeſt, be juſt double to the probability that there is of his coming off an actual winner of $q$, or more, ſtakes at the end of $n$ games; ſuppoſing here, contrary to the propoſition, the duration of the play to be limited to that number.

Note 1. *That when any number of terms, to be taken as in caſe the third, is greater than that of the terms remaining, theſe laſt muſt be uſed inſtead of the former.*

Note 2. *That the ſigns* $+$ *and* $-$ *both prefixed to* $\frac{1}{2}$ *in any expreſſion, are to be uſed, according as that or this ſhall be found neceſſary to make that expreſſion a whole number.*

## EXAMPLE I.

Let $a=1$, $b=1$, $n=11$, and the number of ſtakes each gameſter has to loſe $=3$; then, by caſe the 2d, $1-\frac{\overline{3}|^{\frac{11-1}{2}}}{2^{11-1}}=\frac{781}{1024}$ will be the probability that the play is ended in 11 games: but to find the ſame, according to caſe the 3d, from $1+n+n\times\frac{n-1}{2}$, &c. $(=\overline{1+1}|^{n})=1+11+55+165$, &c. I cut off the 5 $(=\frac{n-p+\frac{q}{2}+\frac{1}{2}}{2})$ firſt terms, take the ſum of the 3 ($q$) laſt of them, as $55+165+330=550$, and reject the reſt; and and becauſe 8 ($=n-p$) is not an odd number, to twice this ſum $=1100$, I add $-330+11$, and divide the aggregate by $2^{11}$, and there comes out $\frac{781}{2048}$ for $\frac{1}{2}$ the value ſought.

## EXAMPLE II.

Let $a=b$, $p=6$, $q=10$, and $n=21$. From $1+21+21\times\frac{20}{2}$, &c. cutting off the firſt 8 terms $1+21+210+1330+5985+20349+54264+116280$, taking their ſum and dividing the ſame by $\overline{2}|^{20}$ we have $\frac{198440}{1048576}=\frac{24805}{131072}$ for the probability of the play ending to the advantage of $A$ in 21 games. And by taking the ſum of the firſt ſix of thoſe terms and dividing as before, there comes out $\frac{27896}{1048576}=\frac{3487}{131072}$ for the probability of the ſame thing happening in favour of $B$; but the ſum of thoſe two is $\frac{7073}{32768}$, and therefore the odds that the play is not ended in 21 games as $32768-7073$ to $7073$, or nearly as 11 to 3.

EXAMPLE

## EXAMPLE III.

Suppoſe $p=3$, $q=4$, $n=11$, $a=2$, and $b=1$; then, according to the 4th caſe, $R$ will be $=a^{11} + 11a^{10}b + 55a^{9}b^{2} + 165a^{8}b^{3} + 330a^{7}b^{4}$, or $2^{7} \times \overline{16+88+220+330+330} = 128 \times 984$, $S=a^{11}=128 \times 16$, $T=0$, &c. $r=b^{11}+11b^{10}a+55b^{9}a^{2}+165b^{8}a^{3}=1+22+220+1320=1563$, $s=0$, &c. and therefore $\dfrac{R - S \times \left.\overline{\frac{b}{a}}\right|^{q} \text{ \&c. } + r \times \left.\overline{\frac{a}{b}}\right|^{p} \text{ \&c.}}{\left.\overline{a+b}\right|^{n}} = \dfrac{138328}{177147}$ is the probability that the play ends to the advantage of $A$ in 11 games; and if to this be added $\frac{6771}{177147}$, ſhewing the like in reſpect to $B$, there will ariſe $\frac{145099}{177147}$ for the required probability of the play being ended in 11 games.

## PROBLEM XXVI.

A *and* B, *whoſe Chances for winning any ſingle Game are in Proportion as* (a) *to* (b), *enter into Play together. What is the Probability that* A *ſhall firſt be a Loſer of* (q) *Stakes, before he is a Winner of* (p) *Stakes, and afterwards a Winner of* (p) *Stakes, and all this in the firſt* (n) *Games?*

## SOLUTION.

It is found in the laſt problem, that the probability which $A$ has of being a winner of $p$ ſtakes in the propoſed number of games, retaining the ſame conſtruction, is

$R—$

$$\frac{R - S\times\left(\frac{b}{a}\right)^{q} + T\times\left(\frac{b}{a}\right)^{q+p} \text{ \&c. } r\times\left(\frac{a}{b}\right)^{p} - s\times\left(\frac{a}{b}\right)^{p+q} \text{ \&c.}}{(a+b)^{n}},$$

when the play is ſuppoſed to terminate upon his loſing $q$ ſtakes, if this ſhould happen before the $n$ games are expired. And it appears by the corollary to the ſame, that the probability he has of being a winner of thoſe ſtakes, in the ſame number of games, when the duration of the play is not reſtrained by what he may happen to loſe, is $R+\frac{rap}{bp}$: therefore by ſo much as this laſt value exceeds the former, by ſo much, it is manifeſt, will the probability be expreſt, that he ſhall be firſt a loſer of $q$ ſtakes, and afterwards a winner of $p$ ſuch, or this exceſs, which is

$$\frac{S\times\left(\frac{b}{a}\right)^{q} - T\times\left(\frac{b}{a}\right)^{q+p} + U\times\left(\frac{b}{a}\right)^{2q+p} \text{ \&c. } + s\times}{(a+b)^{n}}$$

$$\frac{\left(\frac{b}{a}\right)^{p+q} - t\times\left(\frac{b}{a}\right)^{2p+q} + v\times\left(\frac{a}{b}\right)^{2p+2q} \text{ \&c.}}{(a+b)^{n}}$$ will be the value. Q. E. I.

## COROLLARY.

When $a$ is $=b$, the anſwer will be more eaſily had by the following contraction of the foregoing method, *i. e.* From $1+1$ raiſed to the $n$ power, cut off as many terms as there are units in $\frac{n-p+\frac{3}{2}\mp\frac{1}{2}}{2}-q$, and of theſe take the $p$ laſt, reject the $q$ preceding ones, take the $p$ next, reject the $q$ next, &c. Then, if $n-p$ be an odd number, the ſum of all the terms thus taken, divided by $2^{n-1}$, will give the value ſought. And, when $n-p$ is an even number, if the terms of thoſe ſo cut off, whoſe diſtances from the laſt are denoted by $0$, $p$, $p+q$, $2p+q$, &c. be taken with ſigns — and + alternately, and added to twice the ſaid ſum, and the whole be divided by $2^n$, the quotient will be the value ſought in this caſe.

## EXAMPLE.

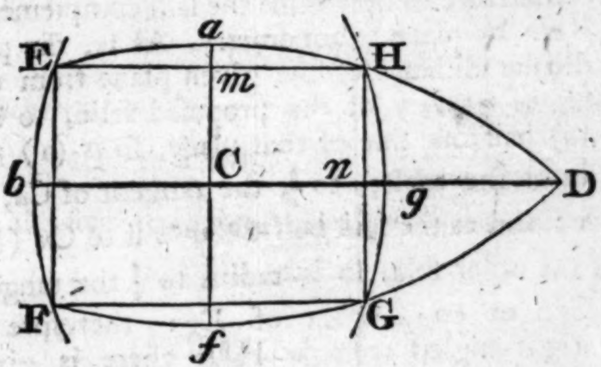
Let $a=b$, $p=5$, $q=3$, and $n=24$. From $1+n+n\times\frac{n-1}{2}$ &c. $(=\overline{1+1}\rceil^{n})$ cutting off the firſt 7 terms, and taking the ſum of the 5 laſt of them, we have 190026, which divided by 4194304 $(=\overline{2}\rceil^{23})$ gives the required value.

## PROBLEM XXVII.

*In a Parallelopipedon, whoſe Sides are to one another in the Ratio of a, b, c; to find at how many Throws any one may undertake that any given Plane,* viz. *ab, may ariſe.*

## SOLUTION.

Imagine the given parallelopipedon to be circumſcribed by a ſpherical ſurface, and to be ſuſtained by a power acting at its center of gravity in the direction of a right line ſtanding perpendicular to the horizon; and while the center of gravity and that line are ſuppoſed to remain at reſt, let the ſolid, together with the circumſcribing ſurface, be ſo moved, that the ſaid perpendicular line may trace out the perimeter, or limits, of the propoſed plane; then it will not be difficult to conceive that that part of the ſpherical ſurface, which will be limited by the interſection, or path deſcribed, by the ſaid perpendicular, through ſuch motion, will be to the whole ſurface, as the probability of the given plane ariſing at any aſſigned throw to unity: it is alſo obvious that the limits of the ſaid part will be arcs of 4 great circles, whoſe chords are the ſides of the given plane. This being premiſed, let the parallelopipedon be now ſuppoſed at reſt, with its upper or given plane, E H G F,



parallel to the horizon; and let the great circles E$a$H, H$g$G, G$f$F, and F$b$E be the limits of the aboveſaid part of the ſpherical ſurface, whoſe content we would find; let alſo 2 other great circles, $bg$, $fa$, biſect the oppoſite arches EF, HG; and EH, FG, and EH and FG be produced to meet $bg$ ſome where as in D: then the

angles

angles C, $b$, $a$, $g$, and $f$ being all right ones, the ſides D$a$, DC, DF, will be each 90 degrees, and C$a$ the meaſure of the angle $a$DC. Wherefore, if (720) twice the degrees in the circle be put to define the content of the whole ſpherical ſurface, C$a$ or its equal HD$g$ will expreſs the content of the triangle $a$DC, and DH$g$+HD$g$ —90°, that of the triangle HD$g$ *(as is proved in* p. 179, *of my Book of* Fluxions.) Whence, by taking the laſt of theſe 2 from the former, we have 90°—$g$HD for the content of the part C$a$H$g$C; wherefore as 720° to 4 times 90°—$g$HD, or as 180° to the complement of the angle $g$HD; ſo is the content of the whole ſurface, to that of the required part EHGFE. Now therefore to determine the ſaid complement; we have by plane trigonometry, As $\frac{1}{2}c$, the perpendicular diſtance of the given plane from the center of gravity of the propoſed ſolid, to C$m$ $(=\frac{1}{2}b)$ half one ſide of that plane, ſo is (1) the radius of the tables, to $\frac{b}{c}$, the tangent of C$a$, or HD$g$; and as the ſaid half-diſtance is to C$n$ $(\frac{1}{2}a)$ half the other ſide, ſo is radius to $\frac{a}{c}$ the tangent of C$g$, or co-tangent of D$g$: therefore in the right-angled triangle HD$g$ there is given $(\frac{b}{c})$ the tangent of the angle D, and $(\frac{a}{c})$ the co-tangent of the ſide D$g$, from whence the ſine of the former and co-ſine of the latter will be eaſily had $\frac{b}{\overline{cc+bb}|^{\frac{1}{2}}}$ and $\frac{a}{\overline{aa+cc}|^{\frac{1}{2}}}$ reſpectively, and then, *per ſpherics*, it will be as $1 : \frac{b}{\overline{cc+bb}|^{\frac{1}{2}}} : : \frac{a}{\overline{aa+cc}|^{\frac{1}{2}}} : \frac{ab}{\overline{aa+cc}|^{\frac{1}{2}} \times \overline{bb+cc}|^{\frac{1}{2}}}$ = the ſine of the complement propoſed to be found.

Now therefore let $A$ be put for the angle whoſe ſine is $\frac{ab}{\overline{aa+cc}|^{\frac{1}{2}} \times \overline{bb+cc}|^{\frac{1}{2}}}$, and $x$ for the required number of throws; then, it follows from what has been ſaid above, that the probability of the given plane ariſing at any aſſigned throw will be $\frac{A}{180}$; therefore that of the contrary being $1-\frac{A}{180}$, we ſhall have $\overline{1-\frac{A}{180}}|^{x}$ equal $\frac{1}{2}$ and conſequently $x$ equal $\frac{-L:2}{L:\overline{1-\frac{A}{180}}}$, or $\frac{124.5}{A} \times \overline{1-\frac{A}{300}}$ nearly. Q. E. I.

## COROLLARY I.

When $a$, $b$, and $c$ are equal, the parallelopipedon becomes a cube, and $\frac{ab}{\overline{aa+cc}|^{\frac{1}{2}} \times \overline{bb+cc}|^{\frac{1}{2}}} = \frac{1}{2}$ = the ſine of 30 degrees; therefore in that caſe $A=30$, and $x = \frac{124.5}{30} \times \overline{1-\frac{30}{360}} = 3.8$.

## COROLLARY II.

If the given plane ($ab$) be but ſmall in reſpect of the others, then the ſine $\frac{ab}{\overline{aa+cc}|^{\frac{1}{2}} \times \overline{bb+cc}|^{\frac{1}{2}}}$ being alſo ſmall, will be nearly as $A$ its correſponding angle, that is, $\frac{ab}{\overline{aa+cc}|^{\frac{1}{2}} \times \overline{bb+cc}|^{\frac{1}{2}}}$ will be to 3.141, &c. as $A$ to 180°; whence $A$ equal $\frac{180ab}{3.1416\times\overline{aa+cc}\rceil^{\frac{1}{2}}\times\overline{bb+cc}\rceil^{\frac{1}{2}}}$ nearly; which value being ſubſtituted in $\frac{124.5}{A}$ (then nearly $=x$) gives $\frac{2.2\times\overline{aa+cc}\rceil^{\frac{1}{2}}\times\overline{bb+cc}\rceil^{\frac{1}{2}}}{ab}$ for the required number of throws in this caſe.

## LEMMA I.

*Suppoſing $A$, $B$, $C$, &c. to be the Coefficients of the Power of $a+x$, whoſe Exponent is $n$; or, $A+Bx+Cx^2+Dx^3$, &c. $=\overline{a+x}\rceil^{n}$: to find the Value of $\frac{A}{1.2.3\ (r)}+\frac{Bx}{2.3.4\ (r)}+\frac{Cx^2}{3.4.5\ (r)}+\frac{Dx^3}{4.5.6\ (r)}+\frac{Ex^4}{5.6.7\ (r)}$, &c. in finite Terms, $r$ being any whole poſitive Number, and $n$ any poſitive Number, or negative one, except an Integer leſs than $r+1$.*

Since $A+Bx+Cx^2+Dx^3$, &c. is $=\overline{a+x}\rceil^{n}$, $A+Bx+Cx^2$, $\times\dot{x}^{r}$ will therefore $=\overline{a+x}\rceil^{n}\times\dot{x}_{r}$; whence, by taking the fluent continually of the laſt equation, we have, Firſt, $\frac{\overline{a+x}\rceil^{n+1}}{n+1}\times\overset{r-1}{\dot{x}}-\frac{a^{n+1}}{n+1}\times\overset{r-1}{\dot{x}}=\overset{r-1}{\dot{x}}$ into $\frac{Ax}{1}+\frac{Bx^2}{2}+\frac{Cx^3}{3}$, &c. Second, $\frac{\overline{a+x}\rceil^{n+2}}{\overline{n+1}\times\overline{n+2}}\times\overset{r-2}{\dot{x}}-\frac{a^{n+2}}{\overline{n+1}\times\overline{n+2}}\times\overset{r-2}{\dot{x}}-\frac{a^{n+1}x\overset{r-2}{\dot{x}}}{n+1}=\overset{r-2}{\dot{x}}$ into $\frac{Ax^2}{1.2}+\frac{Bx^3}{2.3}+\frac{Cx^4}{3.4}$, &c.

And, laſtly, $\frac{\overline{a+x}|^{n+r}}{\overline{n+1}\times\overline{n+2}\ (r)}-\frac{a^{n+r}}{\overline{n+1}\times\overline{n+2}\ (r)}-\frac{a^{n+r-1}x}{\overline{n+1}\times\overline{n+2}\ (r-1)}-\frac{\frac{1}{2}a^{n+r-2}x^{2}}{\overline{n+1}\times\overline{n+2}\ (r-2)}-\frac{\frac{1}{6}a^{n+r-3}x^{3}}{\overline{n+1}\times\overline{n+2}(r-3)}-\frac{\frac{1}{24}a^{n+r-4}x^{4}}{\overline{n+1}\times\overline{n+2}\ (r-4)}\ (r+1)=\frac{Ax^{r}}{1.2.3\ (r)}+\frac{Bx^{r+1}}{2.3.4\ (r)}+\frac{Cx^{r+2}}{3.4.5\ (r)}$, &c. Wherefore, dividing this equation by $x^{r}$, we have $\frac{\overline{a+x}|^{n+r}\times x^{-r}}{\overline{n+1}\times\overline{n+2}\ (r)}-\frac{a^{n+r-1}x^{-r}}{\overline{n+1}\times\overline{n+2}\ (r)}-\frac{a^{n+r-1}x^{-r+1}}{\overline{n+1}\times\overline{n+2}\ (r-1)}-\frac{\frac{1}{2}a^{n+r-2}x^{-r+2}}{\overline{n+1}\times\overline{n+2}\ (r-2)}\ (r+1)=\frac{A}{1.2.3\ (r)}+\frac{Bx}{2.3.4\ (r)}+\frac{Cx^{2}}{3.4.5\ (r)}$, &c. Q. E. I.

## LEMMA II.

*To find the Sum of a Series of Powers whoſe Roots are in arithmetical Progreſſion, as* $d^{n}+\overline{2d}|^{n}+\overline{3d}|^{n}+\overline{4d}|^{n}+\overline{5d}|^{n}$, *&c. continued to x Terms.*

Let $d^{n}$ into $Ax^{n+1}+Bx^{n}+Cx^{n-1}+Dx^{n-2}+Ex^{n-3}$, &c. $+K=d^{n}+\overline{2d}|^{n}+\overline{3d}|^{n}+\overline{4d}|^{n}\ldots\ldots\overline{xd}|^{n}$; then muſt $d^{n}$ into $\overline{Ax+1}|^{n+1}+\overline{Bx+1}|^{n}+\overline{Cx+1}|^{n-1}$, &c. $+K$, it is manifeſt, be $=d^{n}+\overline{2d}|^{n}+\overline{3d}|^{n}+\overline{4d}|^{n}\ldots\ldots\overline{xd}|^{n}+\overline{\overline{x+1}\times d}|^{n}$; wherefore, by taking the former of theſe equations from the latter, and dividing the whole by $d^{n}$, we have $A\times\overline{\overline{x+1}|^{n+1}-x^{n+1}}+B\times\overline{\overline{x+1}|^{n}-x^{n}}+C\times\overline{\overline{x+1}|^{n-1}-x^{n-1}}+D\times\overline{\overline{x+1}|^{n-2}-x^{n-2}}$, &c. $=\overline{x+1}|^{n}$; where expanding the ſeveral powers of $x+1$ in ſeries, and comparing the like terms, we get $A=\frac{1}{n+1}$, $B=\frac{1}{2}$, $C=\frac{n}{3.4}$, $D=0$, $E=\frac{n\times\overline{n-1}\times\overline{n-2}}{2.3.4.5.6}$, $F$ equal 0, $G$ equal $\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}}{2.3.4.5.6.7.6}$, &c. And therefore $d^n+\overline{2d}|^n+\overline{3d}|^n\ldots\ldots\ldots\overline{xd}|^n=\frac{x^{n+1}}{n+1}+\frac{xn}{2}+\frac{nx^{n-1}}{3.4}-\frac{n\times\overline{n-1}\times\overline{n+2}\times x^{n-3}}{2.3.4.5.6}+\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times x^{n-5}}{2.3.4.5.6.7.6}$, &c. $+K$, into $d^n$; which when $x=1$, will become $d^n=\frac{1}{n+1}+\frac{1}{2}+\frac{1}{3.4}-\frac{n\times\overline{n-1}\times\overline{n-2}}{2.3.4.5.6}$, &c. $+K$ in $d^n$: wherefore $K=\frac{-1}{n+1}+\frac{1}{2}-\frac{n}{3.4}+\frac{n\times\overline{n-1}\times\overline{n-2}}{2\ 3.4.5.6}$ &c. conſequently $d^n$ in.

$$\frac{x^{n+1}}{n+1}+\frac{x^n}{2}+\frac{nx^{n-1}}{3.4}-\frac{n\times\overline{n-1}\times\overline{n-2}x^{n-3}}{2.3.4.5.6}+\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times x^{n-5}}{2.3.4.5.6.7.6}-$$

$$\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\overline{n-5}\times\overline{n-6}x^{n-7}}{2.3.4.5.6.7.8.5.6}+$$

$$\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\overline{n-5}\times\overline{n-6}\times\overline{n-7}\times\overline{n-8}x^{n-9}}{2.3.4.5.6.7.8.9.11.12}, \text{ \&c.}$$

$$-\frac{1}{n+1}+\frac{1}{2}-\frac{n}{3.4}+\frac{n\times\overline{n-1}\times\overline{n-2}}{2.3.4.5.6}-\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}}{2.3.4.5.6.7.6}.$$

&c. equal to the value ſought; which when $n$ is a whole poſitive number, will be barely expreſſed by $d^n$ in $\frac{x^{n+1}}{n+1}+\frac{x^n}{2}+\frac{nx^{n-1}}{3.4}-\frac{n\times\overline{n-1}\times\overline{n-2}x^{n-3}}{2.3.4.5.6}$ &c. continued till it terminates, provided that the laſt term thereof when $n$ is an odd number be rejected. Q. E. I.

CORALLARY

## COROLLARY I.

Hence by taking $d=1$, and $n=1, 2, 3, 4,$ &c. ſucceſſively, we ſhall have

$$1+2+3+4+5+6 \ldots\ldots x=\frac{x^2}{2}+\frac{x}{2}$$

$$1^2+2^2+3^2+4^2+5^2 \ldots\ldots x^2=\frac{x^3}{3}+\frac{x^2}{2}+\frac{x}{6}$$

$$1^3+2^3+3^3+4^3+5^3 \ldots\ldots x^3=\frac{x^4}{4}+\frac{x^3}{2}+\frac{x^2}{4}$$

$$1^4+2^4+3^4+4^4+5^4 \ldots\ldots x^4=\frac{x^5}{5}+\frac{x^4}{2}+\frac{x^3}{3}+\frac{x}{30}$$

$$1^5+2^5+3^5+4^5+5^5 \ldots\ldots x^5=\frac{x^6}{6}+\frac{x^5}{2}+\frac{5x^4}{12}-\frac{x^2}{12}$$

&c. &c.

## COROLLARY II.

If $x$ be infinite, and $n$ any poſitive number, or negative one leſs than unity; the firſt term of the ſeries, it is manifeſt, will be infinite in reſpect of the reſt of the terms; and therefore the equation will in that caſe become $1^n+2^n+3^n+4^n \ldots\ldots x^n=\frac{x^{n+1}}{n+1}$.

## LEMMA III.

*To find an Expreſſion or Series, which conſiſting of the ſimple Powers of $x$ and known Coefficients, ſhall be equal to* $1\times2\times3\times4\times5\times6\times7\times8$, &c. *continued to $x$ Factors.*

Pet $P$ denote the given value, and $\overline{x-a}\,L:x, +Ax+B+Cx^{-1}+Dx^{-2}$, &c. the hyp. log. thereof, then will $\overline{x+1-a}\times L:\overline{x+1}, +A\times\overline{x+1}+B+C\times\overline{x+1}^{-1}+D\times\overline{x+1}^{-2}+E\times\overline{x+1}^{-3}$, &c. be $=L:P$

$+L$

$+L:x+1$; from which equation by ſubſtituting the value of $L:P$, &c. there will be $\overline{x-a}\,L:\frac{1+x}{x}+A+C\times\overline{\overline{x+1}^{-1}-x^{-1}}+D\times\overline{\overline{x+1}^{-2}-x^{-2}}+E\times\overline{\overline{x+1}^{-3}-x^{-3}}$, &c. $=0$; this converted into ſimple terms is

$$\left|\begin{array}{cccccccc}
1 & -\frac{x^{-1}}{2} & +\frac{x^{-2}}{3} & -\frac{x^{-3}}{4} & +\frac{x^{-4}}{5} & -\frac{x^{-5}}{6} & +\frac{x^{-6}}{7} & ,\&c. \\
* & -ax^{-1} & +\frac{ax^{-2}}{2} & -\frac{ax^{-3}}{3} & +\frac{ax^{-4}}{4} & -\frac{ax^{-5}}{5} & +\frac{ax^{-6}}{6} & ,\&c. \\
A & -* & -Cx^{-2} & +Cx^{-3} & -Cx^{-4} & +Cx^{-5} & -Cx^{-6} & ,\&c. \\
* & * & * & -2Dx^{-3} & +3Dx^{-4} & -4Dx^{-5} & +5Dx^{-6} & ,\&c. \\
* & * & * & * & -3Ex^{-4} & +6Ex^{-5} & -10Ex^{-6} & ,\&c. \\
* & * & * & * & * & -4Fx^{-5} & +10Fx^{-6} & ,\&c. \\
* & * & * & * & * & * & -5Gx^{-6} & ,\&c. \\
* & & * & & * & & * & \&c.
\end{array}\right|$$

$=0$. Whence $A=-1$, $a=-\frac{1}{2}$, $C=\frac{1}{3.4}$, $D=0$, $E=\frac{-1}{3.4.5.6}$, $F=0$, $G=\frac{1}{5.6.6.7}$, $H=0$, $I=\frac{-1}{5.6.7.8}$, &c. and conſequently $\overline{x+\frac{1}{2}}\times L:x$, $-x+B+\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}-\frac{1}{1680x^7}+\frac{1}{1188x^9}$, &c. $=L:P$. Suppoſe $x=1$, and the equation will become $-1+B+\frac{1}{12}-\frac{1}{360}$, &c. $=0$, and therefore $B$ muſt be $=1-\frac{1}{12}+\frac{1}{360}-\frac{1}{1260}$ &c. which ſubſtituted inſtead thereof, gives $\overline{x+\frac{1}{2}}\times L:x$, $-x+1-\frac{1}{12}+\frac{1}{360}$, &c. $+\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}$ &c. $=L:P$. Now the number whoſe hyp. log. is 1 will be $=2.71828$, &c. that whoſe log. is $\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}$, &c. $=$ $1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}$, &c. and that anſwering to $1-\frac{1}{12}+\frac{1}{360}$, &c. equal to 2.506, &c. which, as Mr. *Demoivre* obſerves from Mr. *Sterling*, is the ſquare root of the periphery of a circle, whoſe ſemi-diameter is unity: wherefore if that periphery be denoted by *c*, and 2.71828, &c. the number whoſe hyp. log. is 1, by *m*, and our logarithmical equation be reduced to one ſhewing the relation of the correſponding numbers, by multiplying together thoſe above found anſwering to the ſeveral parts thereof, we ſhall have $x^{x+\frac{1}{2}}$ in $m^{-x}$ in $c^{\frac{1}{2}}$ in $1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}$, &c. $=P$. Q. E. I.

## COROLLARY.

Becauſe wheu $x$ is a very great number $\frac{1}{12x}+\frac{1}{288x^2}$, &c. becomes inconſiderable, *P* or its equal $1\times2\times3\times4\times5\times6\times7\ldots\times x$ will then be nearly equal to $\left.\overline{\frac{x}{m}}\right|^{x}$ in $\sqrt{cx}$.

## LEMMA IV.

*If c be the Periphery of a Circle, whoſe Radius is Unity, the Ratio of the middle or greateſt Term of the Binomial* $1+1$, *raiſed to a very great Power* $(n)$, *(n), will be to the whole Power, as* 1 *to* $\frac{\sqrt{cn}}{2}$ *very nearly.*

For it is known, from the principles of quadratures, and is proved in page 118 and 121, of my Book of *Fluxions*, that $c$ being, as above ſpecified, the area of the whole curve, whoſe abſciſſa is $x$, and ordinate $\frac{x^n}{\sqrt{1-xx}}$, when $n$ is an even number, will be $\frac{c}{4}$ into $\frac{1}{1}\times\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\times\frac{7}{8} \ldots\ldots\ldots \times\frac{n-1}{n}$, and that the area of the curve, having the ſame abſciſſa, whoſe ordinate is $\frac{x^{n+1}}{\sqrt{1-xx}}$, under the ſame values of $x$ and $n$, is $\frac{1}{1}\times\frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}\ldots\ldots\times\frac{n}{n+1}$. Therefore ſince theſe two areas, when $n$ is very great, are very near equal to each other, $\frac{c}{4}\times\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\times\ldots\ldots\times\frac{n-1}{n}$ will be nearly $=\frac{1}{1}\times\frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}\ldots\times\frac{n}{n+1}$; and therefore, by diviſion, $\frac{c}{4}= \frac{2\times2\times4\times4\times6\times6\ldots\times n\times n}{1\times1\times3\times3\times5\times5\ldots\times\overline{n-1}\times\overline{n-1}\times\overline{n+1}}$ where multiplying both ſides by $n+1$, we get $\frac{c}{4}\times\overline{n+1}$, or $\frac{cn}{4}=\frac{2\times2\times4\times4\ldots\ldots\times n\times n}{1\times1\times3\times3\ldots\ldots\times\overline{n-1}\times\overline{n-1}}$ nearly; the ſquare root whereof is $\frac{\sqrt{cn}}{2}=\frac{4\times4\times6\ldots n}{1\times3\times5\ldots n-1}=$ $\frac{2\times2\times4\times4\ldots n\times n}{1\times2\times3\times4\ldots\overline{n-1}\times n}=2^n\times\frac{1^2\times2^2\times3^2\ldots\ldots\overline{\times\frac{1}{2}n}|^2}{1\times2\times3\times4\ldots\ldots n}=2_n$ $\times\frac{1\times2\times3\times4\ldots\ldots\frac{1}{2}n}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\ldots\ldots\overline{\frac{1}{2}n+1}}$; wherefore it is as

1 :

$1 : \frac{\sqrt{cn}}{2} :: \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \ldots \overline{\frac{1}{2}n+1}}{1 \times 2 \times 3 \times 4 \ldots \frac{1}{2}n} : 2^n$ but the third term of this proportion is known to be equal to the middle term of the binomial propoſed, and the fourth thereof equal to the whole power. Q. E. I.

## LEMMA V.

*To find the Ratio that an aſſigned Term of a Binomial, raiſed to an infinite Power, bears to the whole Series.*

Let $a+b$ be the given binomial, $n$ the index of its power, $r$ the diſtance of the propoſed term from the firſt, and put $s=n-r$: then ſince the value of the ſaid term is $\frac{n \times \overline{n-1} \times \overline{n-2} \ldots \overline{n-r+1} \times a^s b^r}{1.2.3.4.5 \ldots r}$, we ſhall, by equally multiplying by $1 \times 2 \times 3 \times 4 \times 5 \ldots s$, have $\frac{n \times \overline{n-1} \ldots 3 \times 2 \times 1 \times a^s \times b^r}{1.2.3 \ldots r \times 1.2.3 \ldots s}$ which by *Lemma* III. is $= \frac{\overline{\frac{n}{m}}\big|^{n} \times \overline{cn}\big|^{\frac{1}{2}} \times a^s b^r}{\overline{\frac{r}{m}}\big|^{r} \times \overline{cr}\big|^{\frac{1}{2}} \times \overline{\frac{s}{m}}\big|^{s} \times \overline{cs}\big|^{\frac{1}{2}}} = \frac{n^{n+\frac{1}{2}} \times a^s b^r}{r^r \times s^s \times \sqrt{crs}}$; $m$ and $c$ being as there ſpecified; wherefore dividing by $\overline{a+b}\big|^n$ we have $\frac{n^{n+\frac{1}{2}} \times a^s b^r}{r^r \times s^s \times \overline{a+b}\big|^n \times \sqrt{crs}}$.

Q. E. I.

## COROLLARY.

Becauſe the greateſt term of the whole power is that wherein the exponents of $a$ and $b$ are to one another as $a$ to $b$, if $s$ be taken to $r$ in the ratio of $a$ to $b$, or $r=\frac{bn}{a+b}$, and $s=\frac{an}{a+b}$, and theſe values be ſubſtituted as above, we ſhall have $\frac{a+b}{\sqrt{abcn}}$, for the ratio of the greateſt term to the whole power; which therefore, when $a$ and $b$ are equal, will become $\frac{2}{\sqrt{cn}}$ the very ſame as in *Lemma* IV. Hence it is manifeſt, that the value of $c$ in *Lemma* III. is not only near, but exactly equal to the periphery of the circle, whoſe radius is unity; which there was not eaſy to determine.

## LEMMA VI.

*To find the Proportion which the greateſt Term of a Binomial, raiſed to an infinite, or very great Power, bears to a given Number of Terms next it.*

Let $a+b$ be the propoſed binomial, as above, $n$ the index of its power, $y$ the greateſt term, and $p$ the number of terms to be taken on either ſide. Beeauſe the greateſt term is that wherein the indices of $a$ and $b$ are equal to $\frac{na}{a+b}$, and $\frac{nb}{a+b}$; if $\frac{na}{n+b}$ be put $=s$, and $\frac{nb}{a+b}=r$, the next term to it will, it is manifeſt, be $=\frac{s\times by}{\overline{r+1}\times a}$; the next after that $=\frac{s\times\overline{s-1}\times b^2y}{\overline{r+1}\times\overline{r+2}\times a^2}$, the next following $=\frac{s\times\overline{s-1}\times\overline{s-2}\times b^3y}{\overline{r+1}\times\overline{r+2}\times\overline{r+3}\times a^3}$, and conſequently that, whoſe diſtance from the greateſt is $p$, equal to $\frac{s\times\overline{s-1}\times\overline{s-2}\ldots\ldots\overline{s-p+1}}{\overline{r+1}\times\overline{r+2}\times\overline{r+3}\ldots\ldots\overline{r+p}}$ into $\frac{b^p y}{a^p}$; and therefore its hyperbolical logarithm equal to

$$L:y+\left|\begin{array}{r}pL:b\\ pL:s\\ -pL:a\\ -pL:r\end{array}\right|+\left|\begin{array}{l}-\frac{1}{s}-\frac{1}{2s^2}-\frac{1}{3s^3}-\frac{1}{4s^4},\ \&c.\\ -\frac{2}{s}-\frac{4}{2s^2}-\frac{8}{3s^3}-\frac{16}{4s^4},\ \&c.\\ -\frac{3}{s}-\frac{9}{2s^2}-\frac{27}{3s^3}-\frac{81}{4s^4},\ \&c.\\ -\ \&c.\ \text{to}\ p-1\ \text{ſeries}\ldots.\\ -\frac{1}{r}+\frac{1}{2r^2}-\frac{1}{3r^3}+\frac{1}{4r^4},\ \&c.\\ -\frac{2}{r}+\frac{4}{2r^2}-\frac{8}{2r^3}+\frac{16}{4r^4},\ \&c.\\ -\frac{3}{r}+\frac{9}{2r^2}-\frac{27}{3r^3}+\frac{81}{4r^4},\ \&c.\\ -\ \&c.\ \text{to}\ p\ \text{ſeries}\ldots.\end{array}\right.$$

But becauſe $ar$ is $=bs$, $pL:b$, $+pL:s$, $-pL:a$, $-pL:r$ vaniſhes out of the equation. And the numerators of the remaining terms being ſeries of powers, whoſe roots are in arithmetical progreſſion, their ſum will be eaſily had by *Lemma* II. and from thence our expreſſion becomes $L:y$, $\frac{-\overline{p-1}^2-p+1}{2s}$ $\frac{-2\times\overline{p-1}^3-3\times\overline{p-1}^2-p+1}{12s^2}-\frac{\overline{p-1}^4+2\times\overline{p-1}^3+\overline{p-1}^2}{12s^3}$, &c. $-\frac{p^2+p}{2r}-\frac{2p^3+3p^2+p}{12r^2}-\frac{p^4+2p^3+p^2}{12r^3}$, which, $p$ being ſmall in reſpect to $r$ and $s$, will become $=L:y-\frac{pp}{2r}-\frac{pp}{2s}$ very nearly; where by ſubſtituting for $r$ and $s$ their equals $\frac{na}{a+b}$, and $\frac{nb}{a+b}$, we ſhall have $L:y$, $-\frac{pp}{n}\times\frac{\overline{a+b}^2}{2ab}$

$= (L$

$=(L:T)$ the log. of that term whoſe diſtance from the greateſt is denoted by $p$; and therefore $T=y\times\overline{1-\frac{dp^2}{n}+\frac{d^2p^4}{2n^2}-\frac{d^3p^6}{2.3n^3}+\frac{d^4p^8}{2.3.4n^4}}$, &c. $d$, for the ſake of brevity, being put inſtead of $\frac{\overline{a+b}|^2}{2ab}$: hence the ſum of all the terms between the two expreſſed by $y$ and $T$, with the laſt of theſe incluſive, will, by *Lemma* II. be $=yp$ into $1-\frac{dp^2}{3n}+\frac{d^2p^4}{2.5n^2}-\frac{d^3p^6}{2.3.7n^3}+\frac{d^4p^8}{2.3.4.9n^4}-\frac{d^5p^{10}}{2.3.4.5.11n^5}+\frac{d^6p^{12}}{2.3.4.5.6.13n^6}$, *&c.* very nearly. Where, if $v$ be put $=\frac{p^2}{n}$, or $p=\sqrt{vn}$, it will become $y\sqrt{vn}$ into $1-\frac{dv}{3}+\frac{d^2v^2}{2.5}-\frac{d^3v^3}{2.3.7}+\frac{d^4v^4}{2.3.4.9}$, *&c.* from whence the required proportion is manifeſt.

## COROLLARY.

Becauſe the greateſt term divided by the whole ſeries, or $\frac{y}{a+b^n}$, is $=\frac{a+b}{\sqrt{abcn}|}$ (by *Cor.* to *Lem.* 5.) $y$ will be $=\frac{\overline{a+b}|^{n+1}}{\sqrt{abcn}|}$; which being ſubſtituted inſtead thereof in the laſt of the above expreſſions, we ſhall have $\overline{a+b}|^{n+1}\times\sqrt{\frac{v}{abc}}|$ into $1-\frac{dv}{3}+\frac{d^2v^2}{2.5}-\frac{d^3v^3}{2.3.7}+\frac{d^4v^4}{2.3.4.9}$, *&c.* equal to the ſum of as many terms, immediately ſucceeding the greateſt, as there are units in $\sqrt{vn}|$; $c$ being $=2\times3.1415$, *&c.*

and $d=\frac{\overline{a+b}|^{2}}{2ab}$: and therefore when *a* and *b* are equal, the ſaid ſum, it is manifeſt, will become $\overline{a+b}|^{2}\times 2\sqrt{\frac{v}{c}}$ into $1-\frac{2v}{3}+\frac{4v^{2}}{2\cdot5}-\frac{8v^{3}}{2\cdot3\cdot7}+\frac{16v^{4}}{2\cdot3\cdot4\cdot9}-\frac{32v^{5}}{2\cdot3\cdot4\cdot5\cdot11}+\frac{64v^{6}}{2\cdot3\cdot4\cdot5\cdot6\cdot13}$, *&c.*

## PROBLEM XXVIII.

*Suppoſing a given Number,* n+1, *of Adventurers, playing at Raffles, or any other Game of the like Nature, and the firſt of them,* A, *to have raffled and got ſo great a Number, that there are only* (*a*) *Chances for a greater,* (*b*) *Chances for the ſame, and* (*c*) *for a leſſer Number; to find the Probability of his winning: and, ſuppoſing the whole Stake or Thing raffled for, to be in Proportion to the Number of Players; to find alſo what that Number muſt be to make his Advantage by this Circumſtance the greateſt poſſible.*

## SOLUTION.

Since $\frac{b+c}{a+b+c}$ is the probability that no aſſigned player gets a greater number than *A*, the probability that no one of the (*n*) players ſhall bring up a greater number, will be $\frac{\overline{c+b}|^{n}}{\overline{a+b+c}|^{n}}$; or, if $\overline{c+b}|^{n}$ be converted into ſimple terms,

$$\frac{c^{n}+nc^{n-1}b+n\times\frac{n-1}{2}c^{n-2}b^{2}+n\times\frac{n-1}{2}\times\frac{n-2}{3}c^{n-3}b^{3},\ \&c.}{\overline{a+b+c}|^{n}}$$

(by *Cor. Prob.* 1.); the terms of this ſeries being reſpectively equal to the chances, for 0, 1, 2, 3, &c. of thoſe players having an equal number to that of *A*, and all the reſt, at the ſame time, leſſer numbers: wherefore if theſe terms $c^n$, $nc^{n-1}b$, $n\times\frac{n-1}{2}c^{n-2}b^2$, &c. be reſpectively divided by 1, 2, 3, &c. the ſum of the ſeveral quotients applyed to $\overline{a+b+c}|^n$ will be

$$\frac{\frac{c^n}{1}+\frac{nc^{n-1}b}{2}+\frac{n\times\overline{n-1}\times c^{n-2}b^2}{2\cdot3}+\frac{n}{2}\times\frac{n-1}{3}\times\frac{n-2}{4}c^{n-3}b^3\ \&c.}{\overline{a+b+c}|^n}$$

for the firſt part of the anſwer. But this, by *Lemma* 1. is $=\frac{\overline{c+b}|^{n+1}-c^{n+1}}{\overline{n+1}\times b}$ into $\frac{1}{\overline{a+b+c}|^n}$; or, if $a+b+c$ be put $=s$, and $c+b=d$, equal $\frac{d^{n+1}-c^{n+1}}{\overline{n+1}\times s^n b}$; this therefore multipled by $n+1$, the whole ſtake is $\frac{d^{n+1}-c^{n+1}}{bs^n}$, the expectation of *A*, in the ſecond caſe; which by the queſtion muſt be a maximum, and therefore its fluxion $\frac{\dot{n}d^{n+1}}{bs^n}\times L:\frac{d}{s}-\frac{\dot{n}c^{n+1}}{bs^n}\times L:\frac{c}{s}=0$, or $d^{n+1}\times L:\frac{d}{s}=c^{n+1}L:\frac{c}{s}$: wherefore putting the hyperbolic logarithm of $\frac{s}{d}=h$, and that of $\frac{s}{c}=g$, it will be $\overline{\frac{d}{c}}|^{n+1}=\frac{h}{g}$; whence $n+1=\frac{Log.\,h-Log.\,g}{Log.\,d-Log.\,c}$. Q. E. I.

EXAMPLE

## EXAMPLE.

Let *A*'s number be ſuch, that there may be only one chance for a greater, one for the ſame, and 100 for a leſſer number; then $a$ being $=1$, $b=1$, $c=100$, $d=101$, and $s=102$, $g$ will be 0.0198, and $h=0.0985$, and therefore $n+1$, according to the laſt caſe, $=70$; whence it appears that the greateſt expectation that *A* can poſſibly have in the above circumſtance, is, when there are 69 players beſides himſelf; it then being about 25.7 times his own ſtake. But if the players were 100, his expectation, by Caſe 1, would be only 24 times his ſtake, and if they were infinite, it would be indefinitely ſmall. And from hence appears the great diſadvantage that even a good gameſter, or one that has a great number of chances for winning, will have in playing where there are others better than himſelf, even though the greater part of the players are at the ſame time much worſe than himſelf.

## PROBLEM XXIX.

A *and* B, *whoſe Chances for winning any aſſigned Game are in the Proportion of* a, *to* b, *agree to play till* n *ſtakes are won and loſt, on Condition that* A, *at the Beginning of every Game ſhall ſet the Sum* p *to the Sum* $p \times \frac{b}{a}$, *ſo that they may play without Diſadvantage on either Side; it is required to find the preſent Value of all the Winnings that may be betwixt them when the Play is ended.*

## SOLUTION.

Let $s$ be that term of $\overline{a+b}|^{n}$ expreſſed in a ſeries, where the exponent of the power of $a$, divided by that of $b$, is, as near as may be, equal to, but not leſs than $\frac{a}{b}$; and let the diſtance of that term from the firſt, or the index of $b$, in the ſaid term, be denoted by $d$: then becauſe $a^{n}$, $na^{n-1}b$, $n\times\frac{n-1}{2}a^{n-2}b^{2}$, $n\times\frac{n-1}{2}\times\frac{n-2}{3}a^{n-3}b^{3} \ldots s$, the former terms of that ſeries are (as has been found in *Prob.* 5) the reſpective chances for *A*'s winning $n$, $n-1$, $n-2$, $n-3$, *&c.* games preciſely, or gaining the ſums $\frac{pb}{a}\times n$, $\frac{pb}{a}\times\overline{n-1}-p$, $\frac{pb}{a}\times\overline{n-2}-2p$, $\frac{pb}{a}\times\overline{n-3}-3p$; if thoſe terms be reſpectively drawn into theſe ſums, *viz.* $a^{n}$ in $\frac{pb}{a}\times n$, $na^{n-1}b$ in. $\frac{pb}{a}\times\overline{n-1}-p$, &c. the ſeveral products be added together, and the whole divided by $\overline{a+b}|^{n}$, the total number of chances, we ſhall have $\frac{pbs\times\overline{n-d}}{a\times\overline{a+b}|^{n}}$ for half the value ſought; which when the ratio of the above ſaid exponents, is exactly as $a$ to $b$, will become $\frac{pbns}{\overline{a+b}|^{n+1}}$; for then $\overline{n-d}\times b=da$, and $\frac{n-d}{a}=\frac{n}{a+b}$. Q. E. I.

COROLLARY

## COROLLARY II.

Hence, if $n$ be a very great number, $\frac{\text{[illegible]}}{\overline{a+b}|^{n}}$ being $\doteq\frac{a+b}{\sqrt{abcn}}$, by *Lemma* 5. the value in this caſe will be $2p\sqrt{\frac{bn}{ac}}$; $c$ being, as there ſpecified, equal to the periphery of a circle whoſe ſemidiameter is unity.

## EXAMPLE I.

Let $a=2$, $b=1$, $p=1$, and $n=8$; then taking the third term of $a^8+8a^7b+28a^6b^2+56a^5b^3$, *&c.* becauſe the ratio $\frac{5}{3}$ of the exponents in the next term is leſs than $\frac{a}{b}$, there will be $\frac{28a^6b^2\times 6}{\overline{a+b}|^{8}}=1,\frac{1397}{2187}$ for the true value in this caſe.

## EXAMPLE II.

Suppoſe $a=1$, $b=1$, $p=1$, and $n=10000$; and we ſhall have $2p\sqrt{\frac{bn}{ac}}=2\sqrt{\frac{n}{c}}\doteq 79.8$; but if $n$ had been ſuppoſed 1000000, the anſwer would have been only 798.

## PROBLEM XXX.

*Two Gameſters,* A *and* B, *equally ſkilful, enter into Play together, and agree to continue the ſame till* (n) *Games are won and loſt. It is required to* *find the Probability that neither comes off a Winner of* r$\sqrt{n}$ *Stakes, and alſo the Probability than* B *is never a Winner of that Number of Stakes during the whole Time of the Play;* r *being a given, and* n *any very great Number.*

## SOLUTION.

If from 1+1 raiſed to the *n* power be cut off the firſt half of the terms, and as many of the laſt of them be taken as there are units in $\frac{r\sqrt{n}}{2}$ and divided by $\overline{1+1}|^{n}$, the quotient will, by *Prob.* 5, be one half of the value ſought: but the ſaid $\frac{r\sqrt{n}}{2}$ terms according to *Cor.* to *Lem.* 6, by ſubſtituting $\frac{r\sqrt{n}}{2}$ inſtead of $\sqrt{\overline{nv}|}$, will be $\overline{1+1}|^{n} \times \frac{r}{\sqrt{c}}$ in. $1 - \frac{r^2}{3.2} + \frac{r^4}{2.5.4} - \frac{r^6}{2.3.7.8} + \frac{r^8}{2.3.4.9.16} - \frac{r^{10}}{2.3.4.5.11.32} + \frac{r^{12}}{2.3.4.5.6.13.64}$, &c. And therefore the double hereof, divided by $\overline{1+1}|^{n}$ gives $\frac{r^2}{\sqrt{c}}$ in. $1 - \frac{r^2}{3.2} + \frac{r^4}{2.5.4} - \frac{r^6}{2.3.7.8}$, or .798 &c. $\times r$ in. $1 - \frac{r^2}{3.2} + \frac{r^4}{2.5.4} - \frac{r^6}{2.3.7.8} + \frac{r^8}{2.3.4.9.16}$, for the firſt part of the anſwer: but this is alſo the anſwer in the other caſe, as is manifeſt from *Cor.* to *Prob.* 25. Q. E. I.

COROLLARY.

## COROLLARY.

If the probability ſhould be given $= \frac{1}{2}$, and the value of $r$ be required; then it will be $.798$ &c. $\times r$ in: $1 - \frac{r^2}{3.2} + \frac{r^4}{2.5.4} - \frac{r^6}{2.3.7.8}$, &c. $= \frac{1}{2}$; where $r$ will be $=0.674$ &c. Hence it appears that, however great $n$ may be, it is an equal chance that neither of the gameſters comes off a winner of $0.674\sqrt{n}$ ſtakes: and from this and the laſt *Problem* it may be alſo obſerved, that though a gameſter may at ſome times be a very great winner, or, on the other hand a very great loſer, yet, at long-run, there is almoſt any aſſigned odds, if he always plays upon an equality of chance, that there will ſcarcely be any compariſon between his loſs or gain, in the end, and the whole of the money he ventures; and ſecond, that if an event after a great many trials, is continually found to fail or ſucceed more frequently than it ſhould according to the known chances by which it ſeems to be decided, there is the greateſt reaſon to ſuſpect that that event is affected or governed by ſome other cauſe which we are unacquainted with.

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